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Allais Paradoxes Can be Reversed by Presenting Choices in Canonical Split Form*

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Abstract
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1 Introduction

Empirical research has shown that people often violate the implications of expected utility (EU) when choosing between risky alternatives (see Birnbaum, 2008, and Abdellaoui, 2009 for recent surveys). This evidence has motivated the development of alternative theories such as original prospect theory (Kahneman and Tversky, 1979), rank-dependent utility (Quiggin, 1981; 1982; Luce 1991; Luce and Fishburn, 1991), cumulative prospect theory (Starmer and Sugden, 1989; Tversky and Kahneman, 1992; Wakker and Tversky, 1993) and configural weight models such as TAX and RAM (Birnbaum 1997, 1999, 2008). A classical experimental paradigm for observing violations of EU is the constant consequence paradox of Allais (1953). This paradox is usually interpreted as evidence against the "sure thing" principle (Savage, 1954) or independence axiom of EU, but it can be better analyzed as a combination of three simpler assumptions: transitivity, coalescing, and restricted branch independence.

Birnbaum (2004; 2008) disentangled two properties that are usually confounded in tests of the Allais paradoxes (coalescing and restricted branch independence) and concluded that the constant consequence paradoxes are largely due to violations of coalescing, the assumption that if two branches in a gamble lead to the same consequence, they can be combined by adding their probabilities without altering the utility of the gamble. Birnbaum (2004) found that violations of independence, when tested in split form, are actually in the opposite direction from the usual results observed with Allais paradoxes. In other words fanning out of indifference curves (Machina, 1982) seems to be the typical pattern only for coalesced presentation of choices while the opposite pattern of fanning in is more frequent in the split presentation format.

However, Birnbaum (2004) did not analyze the possible role of unequal random errors. Because people do not always make the same decision when presented with the same choice, we need to separate random error from true intention in studies of decision making models (e.g. Camerer, 1989; Starmer and Sugden, 1989; Harless and Camerer, 1994; Hey and Orme, 1994). It has been argued that violations of EU might dissapear when response variability is taken into account (Sopher and Gigliotti, 1993; Schmidt and Hey, 2004; Blavatskyy, 2006; Regenwetter and Stober, 2006; Butler and Loomes, 2007; 2009; Schmidt and Neugebauer, 2007; Birnbaum and Schmidt, 2008; 2009; Berg, Dickhaut, and Rietz, 2009). Because of the importance of Birnbaum's (2004a) conclusion that the constant consequence paradoxes of Allais are due to violations of coalescing, it is important to determine more precisely the extent to which violations attributed to coalescing and to restricted branch independence may be due to random variability in response as opposed to true preference.

Birnbaum, Schmidt, and Schneider (2009) found that violations of EU such as the Allais constant consequence effect cannot be attributed to error, but that violations of independence are reduced markedly by appropriate splitting of branches, when errors are taken into account. In contrast with the findings of Birnbaum (2004a, 2007, 2008) they found only minimal or unsystematic violations of independence in these split forms, when errors are controlled. This paper follows up on these studies by using a design in which all three types of Allais paradoxes are studied in both coalesced and split forms in order to test whether violations of Allais paradoxes independence can be observed in split form when errors are taken into account. Our new findings show that there are indeed strong Allais paradoxes that can be attributed to violations of both coalescing and restricted branch independence. Violations of restricted branch independence are opposite those predicted by cumulative prospect theory (CPT) and opposite the usual Allais paradoxes.

The rest of this paper is organized as follows: The next section defines coalescing and restricted branch independence and shows that if a person satisfies these two properties and transitivity, she or he would show no Allais paradoxes. Section 3 describes the error model; Section 4 presents the TAX model and describes three variants that will be compared by other tests in this paper; Section 5 presents the method of two studies; Section 6 presents the results, which show that violations of EU and CPT cannot be attributed to error, and Section 7 presents our conclusions.

2 Coalescing and the Constant Consequence Paradox of Allais

A *branch* of a gamble is a probability-consequence pair that is distinct in the presentation to the decision maker. For example, B = (x, p; x, q; y, 1 – p – q) is a three branch gamble in which there are two branches leading to consequence x, (x, p) and (x, q), and a third branch, (y, 1 – p – q), leading to y. A = (x, p + q; y, 1 – p – q) is a two-branch gamble in which the two branches are (x, p + q) and (y, 1 – p – q) . In an objective sense, gambles A and B are equivalent; however, people may not treat them as equal.

Coalescing is the assumption that if two branches in a gamble lead to the same consequence, they can be combined by adding their probabilities without altering the utility of the gamble. Thus B = (x, p; x, q; y, 1 – p – q) ~ A = (x, p + q; y, 1 – p – q) and C = (x, p; y, q; y, 1 – p – q) ~ D = (x, p ; y, 1 – p). Violations of coalescing combined with transitivity are sometimes called “splitting effects.” (Birnbaum, Coffey, Mellers, & Weiss, 1992; Starmer & Sugden 1993; Birnbaum & Navarette, 1998 Humphrey, 1995, 2001).

Restricted branch independence is the assumption that if the number of branches and the probability distribution over those branches are the same in both gambles of a choice, the consequence on the common branch can be changed without changing the preference induced by other branches in the same gamble. For three branch gambles, restricted branch independence requires:

$$S = (x, p; y, q; z, 1 - p - q) \succ R = (x', p; y', q; z, 1 - p - q)$$

$$\Leftrightarrow S' = (x, p; y, q; z', 1 - p - q) \succ R' = (x', p; y', q; z', 1 - p - q)$$

Birnbaum (2004) noted that if transitivity is assumed, the constant consequence paradoxes of Allais may be due to either violation of coalescing or to violation of restricted branch independence or to both. He concluded that the cause of the constant consequence paradoxes of Allais is violation of coalescing rather than violation of restricted branch independence. Violations of restricted branch independence actually run in the opposite direction from the Allais paradox. (See also Birnbaum & McIntosh, 1996; Birnbaum & Chavez, 1997; Birnbaum & Navarrete, 1998; Birnbaum, 1999; 2004b; 2007; 2008).

To understand how violations of coalescing may underlie the constant consequence paradoxes of Allais, consider the following two choices. In each case, there are two urns that contain 100 colored marbles with prize values printed on them. The judge decides from which urn a single marble will be drawn randomly, which will determine the prize.

Choice 1: Which do you choose?

A: 20 red marbles to win \$40 80 black marbles to win \$2	B: 10 blue marbles to win \$98 90 yellow marbles to win \$2

Choice 2: Which do you choose?

A′: 80 red marbles to win \$98 20 black marbles to win \$40	B′: 90 blue marbles to win \$98 10 yellow marbles to win \$2

According to EU, people should choose A over B iff they choose A′ over B′. Note that both alternatives have a common branch of .8 to win \$2 in the first choice, and both have a common branch of .8 to win $98 in the second choice. If we cancel the common branches, we see that the two choices are otherwise the same. If people satisfy restricted branch independence, coalescing, and transitivity, then they will agree with EU in the tests of Allais common consequence paradox by choosing A and A′ or B and B′. However, if they violate at least one of these properties, they can violate this independence property. To understand this analysis, note that A = ($40, .2; $2, 0.8) $\succ$ B = ($98, 0.1; $2, 0.9) $\Leftrightarrow$ ($40, .0.1; $40, 0.1; $2, 0.8) $\succ$ ($98, 0.1; $2, 0.1; $2, 0.8) by coalescing and transitivity (we have merely split the higher consequence of A and split the lower consequence of B. Therefore, by restricted branch independence, we can change the consequence on the common branch from ($2, 0.8) to ($98, 0.8), which yields ($98, 0.8; $40, 0.1; $40, 0.1) $\succ$ ($98, 0.8; $98, 0.1; $2, 0.1); by coalescing and transitivity, we have A′ $\succ$ B′.

In Birnbaum's TAX model, however, splitting the higher branch of a gamble makes it better while splitting the branch leading to the lowest consequence of a gamble makes it worse. Note that in this analysis, we have used coalescing four times: we split the higher consequence of A (making it better) and we split the lower consequence of B (making it worse); we then coalesced the lower branch of A (making it better) and coalesced the higher branch of B (making it worse). Therefore, all four coalescing/splitting manipulations all improved A′ relative to B′, according to the TAX model. Thus, even if a person satisfied restricted branch independence, if that person violates coalescing, that person will show the violation of the Allais independence of common consequence property.

According to CPT, people should show no effect of coalescing/splitting (Birnbaum & Navarrete, 1998). Instead, CPT attributes these Allais paradoxes entirely to violations of restricted branch independence. Original prospect theory (OPT) satisfies restricted branch independence but violates coalescing (apart from its editing principles, which imply both coalescing and branch independence due to combination and cancellation). By testing these properties separately, we can test and compare OPT, CPT and EU, and test these three against the predictions of TAX and RAM, which violate both coalescing and branch independence.

The *canonical split form of a choice* is the form in which branches are split so that the corresponding ranked branches in the two gambles have equal probability and the number of branches is minimal. For example, consider the (coalesced) choice between A = ($40, 0.2; $2, 0.8) and B = ($98, 0.1; $2, 0.9); in this case, the choice between C = ($40, .0.1; $40, 0.1; $2, 0.8) and D = ($98, 0.1; $2, 0.1; $2, 0.8) is called the canonical split form of that choice because the splitting results in equal probabilities of corresponding branches and the number of branches is minimal. [Many other split forms of the same choice are possible; for example, E = ($40, .2; $2, 0.1; $2, 0.7) versus F = ($98, 0.05; $98, 0.05; $2, 0.9).]

More generally, we can define three types of Allais independence tests. If any of these properties are violated, we say there is a *constant consequence paradox*.

Let S = (x, p; y, q; z, 1 – p – q) represent a gamble with probability p to win x, q to win y, and otherwise receive z, where $x \geq y \geq z \geq 0$. Let R = (x, p'; y, q'; z, 1 – p' – q'), where $p' > p \geq 0$ and $q' \leq q$. *Type 1 independence* is defined as follows: $S \succ R \Leftrightarrow S' = (x, p; y, q + c; z, 1 - p - q - c) \succ R' = (x, p'; y, q' + c; z, 1 - p' - q' - c)$. *Type 2 independence* assumes the following: $S \succ R \Leftrightarrow S' = (x, p + c; y, q - c; z, 1 - p - q) \succ R' = (x, p' + c; y, q' - c; z, 1 - p' - q')$. *Type 3 independence* assumes: $S \succ R \Leftrightarrow S' = (x, p + c; y, q; z, 1 - p - q - c) \succ R' = (x, p' + c; y, q'; z, 1 - p' - q' - c)$. (The value of c and the other probabilities are constrained so that all probabilities are between 0 and 1) In Type 1, Type 2, and Type 3, a common branch with probability c is shifted from the lowest consequence to the middle consequence, from middle to highest, or from lowest to highest, respectively. Any theory that satisfies transitivity, coalescing, and restricted branch independence will satisfy all three of these properties. EU satisfies all three and is therefore refuted by systematic violations of any of these properties.

By separating tests of coalescing from those of restricted branch independence, we can test among four transitive theories: EU, OPT, CPT, and TAX. CPT and also rank-dependent utility (RDU) imply coalescing but violate restricted branch independence. Therefore, RDU and CPT models imply that there should be no effect of presenting choices in coalesced or in a split form. Original prospect theory (OPT) violates coalescing but satisfies restricted branch independence. TAX violates both coalescing and restricted branch independence, but it violates restricted branch independence in the opposite direction from that of prospect theories.

In this study, each of six choices (Figure 1) will be presented in both coalesced and canonical split form. This allows tests comparing EU, OPT, CPT, and TAX. In addition, this paper will analyze these paradoxes while controlling for random errors by means of the model described in the next section.

3. Error Model

Because the same person may make different decisions when faced with the same choice, it might happen that a person could show violations of certain properties simply due to random variability in response.

Consider the case of one choice problem presented twice, for example, between S and R. There are four response patterns possible, RR, RS, SR, and SS. According to the true and error model, the probability that a person will show the RS pattern is given as follows:

(1) $P(RS) = p(1 - e) + (1 - p)e = e(1 - e)$,

where p is the true probability of preferring R and e is the error rate on this choice. This equation shows that we can use the variability of response to the same problem to estimate the error rate, e. Note that P(SR) should also equal $e(1 - e) = P(RS)$, which means that this model can be tested. In addition, $P(RR) = p(1 - e)(1 - e) + (1 - p)ee$, and $P(SS) = pee + (1 - p)(1 - e)(1 - e)$.

Now consider a two-choice problem such as a test of the Allais paradox between S and R and between S′ and R′. The probability that a person shows the observed preference pattern RS′ is given as follows:

(2) $P(RS') = pRR'(1 - e)e' + pRS'(1 - e)(1 - e') + pSR'ee' + pSS'e(1 - e')$

In this expression, P(RS′) is the probability of exhibiting this preference pattern; e is the error rate in the choice between R and S, and e′ is the error rate in the choice between R′ and S′. This probability is the sum of four terms, each representing the probability of having one of the "true" patterns (pRR′, pRS′, pSR′, and pSS′) and with the appropriate pattern of errors and correct responses to produce each observed data pattern given that true pattern. For example, the person who truly has the RR′ pattern could produce the RS′ pattern by correctly reporting the first choice and making an "error" on the second choice. There are three other equations like (2), each showing the probability of an observed data pattern given the model.

As noted by Birnbaum and Schmidt (2008), this model has been applied by Sopher and Gigliotti (1993) with the assumption that error rates are unequal but transitivity is satisfied. The same data would refute transitivity if the error rates were assumed equal, as assumed by Harless and Camerer (1994). Thus, the conclusion one reaches depends on the specification of the errors. In order to reach stronger conclusions concerning Allais paradoxes, we need a way to estimate parameters that do not assume that error rates are necessarily equal or that EU is correct. Put another way, we need to enrich the structure of the data so that we can determine the error rates without assuming a utility model. As shown in Equation 1, this can be done by adding repetitions to presentations of each choice in a test of transitivity or of the other tests of EU or CPT.

We use the term *general model* to refer to the model in which all four true probabilities are allowed to be non-zero. The Chi-Square statistic is defined as follows:

(3) $\chi 2(df) = \Sigma(fi - qi)2/qi$

where fi is the observed frequency, qi is the predicted frequency of a particular response pattern, and df are degrees of freedom. Parameters are selected to minimize this statistic. Without replicates, there are only four cell frequencies, SS′, SR′, RS′, and RR′, with two error rates and four "true" probabilities to estimate. In that case, the model is underdetermined; many solutions will fit the data equally well and it is not possible to determine whether particular violations of a model are due to random error or true preference.

However, with two choices and two replications, there are 2 X 2 X 2 X 2 = 16 cells in Equation 3. These 16 cells have 15 degrees of freedom because they sum to the number of participants. With replicated data, therefore, the model is over-determined. There are two error terms to estimate and four "true" probabilities (which use 3 df because they sum to 1), so there are 6 parameters to estimate using 5 degrees of freedom, leaving 10 degrees of freedom to test the general true and error model. The EU theory is then a special case of this general model in which two of the true probabilities are fixed to zero (pSR′ = pSR′ = 0). When S versus R and S′ versus R′ are a choice presented in coalesced and split form, respectively, then both CPT and EU are special cases in which pSR′ = pSR′ = 0.

The difference in $\chi 2$ between a fit of the model that allows all four true patterns to have non-zero probabilities and the special case in which pRS′ = pSR′ = 0 is also chi-square distributed with 2 degrees of freedom. This test evaluates whether observed deviations from a model are significant or whether they might be attributable to random errors in response. In sum, by the use of replications we can estimate the error terms and test the applicability of EU, OPT, CPT, and compare them with TAX and RAM.

4. TAX Model and Other Tests

Birnbaum and Stegner (1979, Equation 10) proposed a "revised" configural weight model in which configural weights are transferred in proportion to the absolute weight of stimulus that loses weight. Consider gambles defined as G = (x_1, p_1; x_2,p_2; ... x_j, p_j; ... x_i, p_i;…; x_n, p_n), where the outcomes are ordered such that $x_1 > x_2 > ... > x_j > ... x_i > ...> x_n > 0$ and $\Sigma\ p_j = 1$. Note that $x_j > x_i$. This model can be written as follows:

(3)
$$U(G) = \frac{\sum_{i=1}^{n} t(p_i)u(x_i) + \sum_{i=2}^{n}\sum_{j=1}^{i-1}[u(x_i) - u(x_j)]\omega(p_i, p_j, n)}{\sum_{i=1}^{n} t(p_i)}$$

where U(G) is the utility of the gamble, t(p) is a weighting function of probability, u(x) is the utility function of money, and ω(pi, pj) is the configural transfer of weight between outcomes x_i and x_j. Note that if the configural transfer is positive, then the branch with a lower valued consequence transfers weight to the branch with the higher valued consequence; when it is negative, then the higher-valued outcome loses weight and the lower valued outcome gains this same weight. Birnbaum and Chavez (1997) proposed the following assumptions concerning the weight transfers:

(4)
$$\omega(p_i, p_j, n) = \left| \begin{array}{ll} \delta t(p_j)/(n+1), & \delta \geq 0 \\ \delta t(p_i)/(n+1), & \delta < 0 \end{array} \right.$$

where n is the number of distinct outcomes. If $\delta = 1$, then the weights of two, three, and four equally likely outcomes will be (2/3, 1/3), (3/6, 2/6, 1/6), and (.4, .3, .2, .1), for lowest to highest outcomes, respectively, consistent with the model of Birnbaum and McIntosh (1996). For small consequences, 0 < x < $150, it has been found that u(x) = x provides a reasonable approximation to choice data from undergraduates. In addition the weighting function of probability has been approximated as $t(p) = p^{0.7}$. This model is known as the “prior” TAX model because these parameters have been used to predict new experiments, such as those of

Birnbaum (2004a, 2005, 2007, 2008; Birnbaum & Bahra, 2005). [The notation in Equations 3 and 4 agrees with that of Birnbaum and Chavez (1997), except that a different notational convention has been selected, such that the items are ranked from greatest to least; consequently, $\delta > 0$ here corresponds to $\delta < 0$ in Birnbaum and Chavez (1997)].

According to the TAX model, splitting the upper branch of a gamble should make the gamble better and splitting the lower branch of a gamble should make a gamble worse (Birnbaum, 2008). This model thus violates both coalescing and branch-splitting independence (Birnbaum, 2007). The present studies test a new property not yet tested in previous research; namely, the effect of splitting both upper and lower branches of a gamble in the same way. For example, consider A = ($98, 0.5; $2, 0.5) and B = ($98, 0.25; $98, 0.25; $2, 0.25; $2, 0.25). B is the same as A except for coalescing. We call this manipulation "double splitting" because both upper and lower branches are split in the same way. According to the prior TAX model the overall utility of a gamble tends to decrease as the number of branches in such double splitting increases, consistent with the idea that increasing complexity of an alternative lowers its evaluation (Sonsino, Ben-Zion, & Mador, 2002). This manipulation allows us to compare three variants of the TAX model in which the term $(n + 1)$ is replaced by (n + k), where k > 0, k < 0 or k = 0, which imply that double splitting will decrease, increase, or have no effect on the utility of a gamble, respectively.

5. Method

Participants made choices between gambles, which were viewed via computers in a lab. Each choice appeared as in the following example:

```
7. Which do you choose?
          M: 10 blue marbles to win $40
             10 green marbles to win $40
             80 black marbles to win $2
      OR
          N: 10 red marbles to win $98
```

```
10 white marbles to win $2
80 black marbles to win $2
```

Instructions read (in part) as follows:

"Your prize will be determined by the color of marble that is drawn randomly from an urn. The urns always have exactly 100 marbles, so the number of any given color tells you the percentage that win a given prize." They were instructed to click a button beside the gamble they would rather play in each choice. They were informed that 3 lucky participants would be selected at random to play one of their chosen gambles for real money, with prizes as high as $100, so they should choose carefully. Prizes were awarded as promised.

The main design of Studies 1 and 2 was employed to test the extent to which the Allais paradoxes are produced by violations of restricted branch independence or by violations of coalescing while controlling for error. Figure 1 shows the choices used in the main design. This triangular diagram represents a probability simplex depicting all possible lotteries based upon the three consequences: $98, $40, and $2. The horizontal axis gives the probability to win $2 and the vertical axis the probability to win $98. The probability of $40 in a given lottery equals 1 minus the sum of these probabilities. A "sure thing" to win $40, $98, or $2 correspond to the lower left corner, upper left corner, and lower right corner of this figure, respectively.

Each line segment in Figure 1 represents a choice between a "safe" gamble and a "risky" gamble. All six choices should yield the same preference according to EU, i.e. a person should choose always the risky or always the safe alternative. Each of these six choices was presented in both coalesced and in canonical split form. In Study 2, the same design was used but consequences $98, $40, and $2 were changed to $96, $48, and $4, respectively.

Insert Figure1 about here.

Table 1 lists the choices of Figure 1 in both coalesced and split form. Each choice in Table 1 is created from the choice in the row above by either coalescing/splitting or by applying restricted branch independence. People should make the same choice in Table 1 in every row, according to EU, except for random error.

In Choice 1 of Table 1 (first row), the common branch is 80 marbles to win \$2 (Choice 1), 80 to win \$40 (Choice 2), or 80 to win \$98 (Choice 3). Choice 4 is generated from Choice 1 by changing 40 marbles to win \$2 to 40 to win \$40; Choice 5 is generated from Choice 4 by changing the common branch of 40 marbles to win \$2 to 40 marbles to win \$98. Choice 6 is generated from Choice 1 by changing a common branch of 40 marbles to win \$2 to 40 marbles to win \$98.

Choices marked 1a, 2a, 3a, etc. are the same as Choices 1, 2, 3, etc, respectively, except these are presented in canonical split form. For example, Choice 1a is the same as Choice 1, except the branch of 20 marbles to win \$40 in the safe gamble has been split to two branches of 10 marbles each to win \$40, and the branch of 90 marbles to win \$2 in the risky gamble has been split into a branch with 80 marbles to win \$2 and 10 to win \$2. Note that both gambles in Choice 1a have the same number of marbles on corresponding ranked branches. Choice 6c is another split form of Choice 6, designed to improve the "safe" gamble relative to the "risky" one.

Choices 1, 2, and 3 of Study 1 match choices studied by Birnbaum (2004a, Series A).

Insert Tables 1 and 2 about here.

The first two choices served as a "warm-up," and there were 11 additional choices shown in Table 2, which were presented on even numbered trials. These additional trials tested coalescing and double splitting of gambles into different numbers of branches. Complete materials can be viewed at URLs:

http://psych.fullerton.edu/mbirnbaum/decisions/allais5_.htm

http://psych.fullerton.edu/mbirnbaum/decisions/allais6_.htm

Participants were 211 undergraduates enrolled psychology department of California State University, Fullerton (USA). There were 104 and 107 who served in Studies 1 and 2, respectively. There were 57% and 62% females, and 87% and 86% were 20 years of age or younger in the two groups, respectively. Each person completed the task twice, separated by other tasks that required about 20 minutes intervening time.

6. Results and Discussion

The choice proportions in the two studies are shown in Columns 5 and 6 of Tables 1 and 2, with the estimated values of true *p* and *e*, based on the true and error model applied to replications of each choice. The two studies produced results that are very similar. The first six rows in Table 1 match choices used by Birnbaum (2004a). Chi-square tests of fit for the true and error model were conducted for each choice in each study. The median value of these Chi-squares was 0.95; only 4 of the 52 tests exceeded 3.84, which is the critical value at the α = 0.05 level of significance (three were significant in Study 1 and one in Study 2). Apparently, the true and error model can be retained for this aspect of the data.

According to EU, a person should choose either the “risky” gamble or the “safe” gamble in each choice of Table 1, but should not switch, except by random error. Instead, the estimated “true” choice proportions vary from as low as 0.09 to as high as 0.80, depending on the common consequence and form of presentation (split or coalesced).

According to RDU, CPT, and other models that satisfy coalescing, choices should not depend on the form of presentation (split versus coalesced). Instead, there are large effects of splitting that show up in both observed and estimated “true” choice proportions. Choice type 1 versus 1a, 2 versus 2a, 3 versus 3a, and 4 versus 4a all produce splitting effects large enough to reverse the modal “true” choice. Furthermore, there are violations of restricted branch independence; note that Choices 1a, 4a, 5a, 6a, and 6c all yield “true” proportions less than 0.5, but Choices 2a and 3a yield estimated choice proportions that exceed 0.5.

The largest shift in choice proportions is between Choice type 6 and Choice type 4 (Choices No. 9 and 15). In both of cases, the choice is between a 2-branch and a 3-branch gamble in which the additional branch is the intermediate consequence. In both cases, people prefer the gamble with the additional branch.

The prior TAX model correctly predicts the modal choices in all nine cases of Table 1 but it fails to predict the modal response in Choices #13, #26, #15, and #3 (Choice types 2, 2a, 6, and 6a). In all four of these cases, people tend to prefer the gamble that gives three possible outcomes over the gamble giving only one or two possible consequences. Birnbaum (2004a, 2007) found that a slight majority preferred the sure thing in Choice type 2 or the split was nearly equal, except when prizes were very large (Birnbaum, 2007, p. 161), as in the original versions of the Allais paradox. With large prizes, most people preferred the sure thing. CPT with its prior parameters also fails to predict these four choices in Table 1, and it also fails to predict four other choices that were correctly predicted by prior TAX: #7, 23, 5, and 17. All four of these cases involve splitting (Choice types 1a, 3a, 4a, and 6c).

Table 2 shows that double splitting of both the upper and lower branches of a gamble did not change the choice proportions by very much. This suggests that the proportion of weight transferred in the prior TAX model should be written as δ/n rather than $\delta/(n + 1)$. A caveat is that with so many choices using this splitting manipulation in the study, participants may have learned to recognize the equivalence of the double splitting manipulation; therefore, these results might not generalize to a study in which fewer choices of this type are presented.

Choice 20 involves a test of first order stochastic dominance of a type that has yielded about 70% observed violations with undergraduates (Birnbaum, 2007, 2008; Birnbaum & Navarrete, 1998). In this study, the percentage of violations is still significantly above 50%, but somewhat lower in Problem 20 (60 and 61% in Studies 1 and 2, respectively). Choice 4 is a test of splitting related to the test of stochastic dominance in Choice 20. The prior TAX model correctly predicted these two modal choices.

Table 3 presents detailed tests of the RDU/CPT family of models. These models imply that there should be no violations of coalescing; that is, there should be no splitting effects. Data were partitioned into the frequencies of showing each data pattern in one repetition or the other but not both and the number who showed each pattern on both repetitions. These eight frequencies sum to the number of participants, so there are 7 degrees of freedom in the data. From these are estimated the true proportions of participants who showed each pattern. These sum to 1, so there are at most 3 degrees of freedom to estimate from these data. In most cases, however, one of the cells can be set to zero without loss of fit. The values shown are for the simplest model for which deviations are not significant.

These proportions in Table 3 provide estimates of the "true" proportions of participants who had each choice pattern. For example, for choices 19 by 23, the estimate is that 47% of participants had the true choice pattern *SR';* that is, they preferred the "safe" gamble in Choice 19 (coalesced form) and preferred the "risky" gamble in Choice 23 (split form, where the branch leading to the best outcome of *R'* has been split). The test of fit of the true and error model in this case has a Chi-square of 1.99 with 5 degrees of freedom, which is not significant. However, when the parameters representing true proportions of *SR'* and *RS'* were both fixed to zero, the Chi-square increased to 127.48, a difference of 125.49 for one degrees of freedom. The critical value of Chi-square with two degrees of freedom and $\alpha = .01$ is 9.21, so this value (and all other values in Table 3) far exceed the threshold for significance. Examining the pattern of splitting effects, in every case in which the largest consequence of one gamble is split, that gamble is more likely chosen in the split form than in the coalesced form. This pattern is consistent with the TAX model.

In choices 13 by 26, a “sure thing” to win $40 is split into three branches leading to $40. This trial is a simple test of idempotence which demands that splitting a one-branch gamble (i.e. a sure outcome) into several branches should not influence choice behavior. Idempotence is implied by EU, CPT, RDU, and TAX, but not by models of Meginniss (1976) and Luce, et al. (in press). The overall choice proportions in Table 1 are largely unaffected by this manipulation; however, the analysis of choice combinations in Table 3 shows that an approximately equal number of participants were affected in opposite ways by this manipulation. In both studies, the statistical analysis indicated that significant numbers of people shifted in opposite ways.

According to EU, all choices in Table 1 should give the same result, apart from error. Because there are 13 choices in Table 1, there are (13 X 12)/2 = 78 possible tests of EU that can be analyzed, many of which are redundant. Table 4 presents tests of two Allais paradoxes that constitute the largest effects in the tables that do not involve the same choices. According to the analyses, 48% and 67% of participants show reversals in Choices 21 by 19 and Choices 9 by 15 in Study 1 and the figures are 65% and 74% in Study 2, respectively. The true and error model gives an acceptable fit to the data but EU can be rejected with all four Chi-squares exceeding 300. These results show that Allais paradoxes are substantial in coalesced form, even when errors are taken into account.

The last four rows of Table 4 show the same choices presented in split form. These choices test the property of restricted branch independence. According to OPT as well as EU, restricted branch independence should be satisfied. According to CPT, the split forms should be exactly as the coalesced forms, apart from error. However, these results are quite different from those in the coalesced form and they do not satisfy RBI. Three of the four tests show significant violations of RBI. In both cases of Choices 7 by 23, there are significant violation of RBI that are opposite the pattern observed in the coalesced forms. In this case there are 25% and 17% of the sample that showed the *SR′* pattern in Studies 2 and 3, respectively, whereas the data in the coalesced form are consistent with the hypothesis that no one showed this pattern.

7. Conclusions

In this study we observed substantial violations of both restricted branch independence and coalescing that cannot be attributed to errors. These findings refute expected utility as well as original and cumulative prospect theory. Our results confirm and extend those of Birnbaum (2004a; 2007) and Birnbaum, Schmidt, and Schneider (2009). Whereas Birnbaum, et al. (2009) found minimal violations of restricted branch independence in the split form, these data show convincing evidence that this property is violated in the opposite manner from what is observed in the Allais paradoxes in coalesced form and opposite of the pattern predicted by CPT. In particular the fact that typical Allais constant consequence paradoxes are reversed in the split presentation shows that it should be unsustainable to ignore this issue as has been done by many studies of this topic. An interesting question for future research is whether violations of EU such as the constant ratio paradox are affected by the form of presentation to a similar extent.

The TAX model gives a fairly good description of the effects of splitting and of the violations of restricted branch independence. However, this model did not correctly predict the choices in cases in which the number of outcomes differed between the two gambles of a choice. In addition, the prior TAX model implies that double splitting should diminish the utility of a gamble but participants on average were not much affected by this manipulation.

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Table 1. Choices used in Study 1. Choices were the same in Study 2, except the values were adjusted slightly. According to EU theory, all choice proportions should be equal in this table. Choice Type refers to Figure 1.

Problem		Choice		% R Study 1	% R Study 2	True & Error Model—Study 1		True & Error Model Study 2	
No	type	*S*	*R*	N=104	N=107	*p*	*e*	*P*	*e*
21	1	20 to win $40 80 to win $2	10 to win $98 90 to win $2	0.59	0.66	0.64	0.17	0.69	0.11
7	1a	10 to win $40 10 to win $40 80 to win $2	10 to win $98 10 to win $2 80 to win $2	0.42	0.40	0.40	0.11	0.37	0.11
13	2	win $40 for sure	10 to win $98 80 to win $40 10 to win $2	0.57	0.63	0.60	0.13	0.61	0.10
26	2a	10 to win $40 80 to win $40 10 to win $40	10 to win $98 80 to win $40 10 to win $2	0.59	0.56	0.61	0.09	0.59	0.12
19	3	80 to win $98 20 to win $40	90 to win $98 10 to win $2	0.25	0.17	0.16	0.15	0.09	0.09
23	3a	80 to win $98 10 to win $40 10 to win $40	80 to win $98 10 to win $98 10 to win $2	0.60	0.47	0.62	0.23	0.53	0.14
9	4	60 to win $40 40 to win $2	10 to win $98 40 to win $40 50 to win $2	0.73	0.74	0.80	0.13	0.88	0.19
5	4a	10 to win $40 40 to win $40 10 to win $40 40 to win $2	10 to win $98 40 to win $40 10 to win $2 40 to win $2	0.42	0.46	0.38	0.19	0.44	0.13
25	5	40 to win $98 60 to win $40	50 to win $98 40 to win $40 10 to win $2	0.46	0.41	0.43	0.17	0.40	0.19
11	5a	40 to win $98 10 to win $40 40 to win $40 10 to win $40	40 to win $98 10 to win $98 40 to win $40 10 to win $2	0.46	0.40	0.38	0.09	0.38	0.09
15	6	40 to win $98 20 to win $40 40 to win $2	50 to win $98 50 to win $2	0.21	0.24	0.09	0.15	0.10	0.16
3	6a	40 to win $98 10 to win $40 10 to win $40 40 to win $2	40 to win $98 10 to win $98 10 to win $2 40 to win $2	0.40	0.34	0.34	0.19	0.29	0.10
17	6c	10 to win $98 30 to win $98 20 to win $40 40 to win $2	50 to win $98 30 to win $2 10 to win $2 10 to win $2	0.23	0.20	0.17	0.08	0.16	0.15

Table 2. Design testing double splitting. Trial 20 tests stochastic dominance. 12 and 8 test idempotence. In Study 2, ($98, $40, $2) were replaced by ($96, $44, $4).

Problem		Choice		% R Study 1	% R Study 2	True & Error Study 1		True & Error Study 2	
No	type	*R*	*S*	N=104	N=107	*p*	*e*	*p*	*e*
1	7	50 to win $98 50 to win $2	$40 for sure	0.64	0.63	0.66	0.06	0.65	0.05
14	7a	25 to win $98 25 to win $98 25 to win $2 25 to win $2	$40 for sure	0.60	0.62	0.60	0.02	0.61	0.04
16	7b	13 to win $98 13 to win $98 12 to win $98 12 to win $98 12 to win $2 12 to win $2 13 to win $2 13 to win $2	$40 for sure	0.60	0.62	0.61	0.04	0.61	0.04
12	7c	25 to win $98 25 to win $98 25 to win $2 25 to win $2	25 to win $40 25 to win $40 25 to win $40 25 to win $40	0.65	0.66	0.66	0.03	0.67	0.05
2	8	50 to win $98 50 to win $2	$48 for sure	0.68	0.64	0.70	0.06	0.69	0.04
6	8a	25 to win $98 25 to win $98 25 to win $2 25 to win $2	$48 for sure	0.60	0.63	0.60	0.03	0.63	0.02
24	8b	13 to win $98 13 to win $98 12 to win $98 12 to win $98 12 to win $2 12 to win $2 13 to win $2 13 to win $2	$48 for sure	0.61	0.62	0.61	0.04	0.62	0.04
8	8c	25 to win $98 25 to win $98 25 to win $2 25 to win $2	25 to win $48 25 to win $48 25 to win $48 25 to win $48	0.65	0.67	0.66	0.06	0.69	0.04
22	9	70 to win $98 30 to win $2	$50 for sure	0.41	0.43	0.40	0.02	0.42	0.04
18	9a	70 to win $98 10 to win $2 10 to win $2 10 to win $2	$50 for sure	0.46	0.43	0.45	0.07	0.44	0.04
10	9b	50 to win $98 10 to win $98 10 to win $98 30 to win $2	$50 for sure	0.33	0.36	0.31	0.05	0.35	0.08
4	split	90 to win $98 05 to win $2 05 to win $2	85 to win $98 05 to win $98 10 to win $2	0.56	0.58	0.59	0.14	0.59	0.12
20	SD	90 to win $98 05 to win $4 05 to win $2	85 to win $98 05 to win $96 10 to win $2	0.60	0.61	0.65	0.21	0.66	0.15

Table 3. Tests of Splitting Effects. All splitting effects are consistent with the description that splitting the highest consequence improves the gamble. Choice pair 13 by 26 is a test of idempotence. Splitting the "sure thing" to win \$40 into three branches (all to win \$40) improves it for some people and makes it worse for others.

		Patterns				Fit (χ^2)	
Study	Choice No	*SS'*	*SR'*	*RS'*	*RR'*	T&E	CPT
1	21 by 7	0.42	0.00	0.14	0.44	5.18	27.30
	19 by 23	0.35	0.47	0.00	0.18	1.99	125.49
	9 by 5	0.23	0.00	0.34	0.43	3.22	138.67
	15 by 3	0.72	0.17	0.00	0.12	3.52	19.54
	3 by 17	0.68	0.00	0.14	0.18	0.37	21.44
	13 by 26	0.30	0.11	0.09	0.50	0.53	80.19 (2)
2	21 by 7	0.31	0.00	0.29	0.40	3.68	280.45
	19 by 23	0.58	0.28	0.00	0.14	5.28	215.14
	9 by 5	0.17	0.00	0.34	0.49	5.93	139.10
	15 by 3	0.72	0.11	0.00	0.17	4.87	33.70
	3 by 17	0.74	0.00	0.12	0.14	6.28	49.69
	13 by 26	0.32	0.03	0.08	0.56	3.75	34.45 (2)

Table 4. Tests of EU model. Choices 21 by 19 and 9 by 15 test Allais paradoxes in the coalesced form. Choices 7 by 23 and 5 by 3 test the same paradoxes in canonical split form, which tests restricted branch independence. EU implies that both tests should be satisfied.

		Patterns				Fit (χ^2)	
Study	Choice No	*SS'*	*SR'*	*RS'*	*RR'*	T&E	EU
1	21 by 19	0.35	0.00	0.48	0.17	2.01	300.39
	9 by 15	0.20	0.00	0.67	0.13	2.89	730.64
2	21 by 19	0.26	0.03	0.65	0.06	0.91	1612.40
	9 by 15	0.09	0.00	0.74	0.17	2.79	521.05
1	7 by 23	0.31	0.25	0.06	0.38	9.25	31.87
	5 by 3	0.64	0.00	0.00	0.36	2.31	(0)
2	7 by 23	0.45	0.17	0.11	0.27	0.83	113.90
	5 by 3	0.60	0.00	0.07	0.33	3.24	19.08

Figure 1. Design of the choices in the main design. In Study 1 the consequences were ($98, $40, and $2). In Study 2 these were changed to ($96, $48, and $4). In addition, each choice was presented in canonical split form.