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Advances in Applied Mathematics 34 (2005) 295–312

ADVANCES IN Applied Mathematics

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Linear extensions of ranked posets, enumerated by descents. A problem of Stanley from the 1981 Banff Conference on Ordered Sets

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Received 4 November 2003; accepted 16 May 2004

Available online 21 November 2004

Abstract

Let P be a naturally labelled, ranked (graded) poset of rank r and cardinality n. Let H_k be the set of linear extensions of P with k descents. An explicit bijection between H_k and $H_{n-1-r-k}$ is constructed using the involution principle ($0 \leqslant k \leqslant n-1-r$). A problem of Richard P. Stanley from 1981 is thereby solved.

Keywords: (Partially) Ordered set; Linear extension; Natural labelling; Descent; h-vector; P-partition; Eulerian numbers

1. The five of hearts

Major Percy MacMahon, that great British combinatorialist of the turn of the last century, proved the following result in his classic *Combinatory Analysis* [11, Section IV, Chapter V, Sect. 179–180, pp. 212–213].

Take m different numbers (say, the integers 1 through m), each number repeated $r+1$ times, so that there are $n = m(r+1)$ numbers in all. Consider all possible ways of listing these n numbers in a row; if $r = 0$, we are just listing all possible permutations of m objects. (Knuth uses the analogy of shuffling a deck of cards, where suit is ignored: in this case,

0196-8858/$ – see front matter
doi:10.1016/j.aam.2004.05.007

Table 1.1
MacMahon's theorem for $m = 2$ and $r = 2$

H_0	H_1	H_2	H_3
111222	112122	221211	212121
	112212	212211	
	112221	212112	
	121122	221121	
	122112	211221	
	122211	211212	
	211122	122121	
	221112	121221	
	222111	121212	

$m = 13$ and $r = 3$ [9, p. 43].) For each listing, count the number of "descents," the number of places where a bigger number *immediately* precedes a smaller number.

For instance, if $m = 2$ and $r = 2$, there are 20 possibilities (see Table 1.1).

Let H_k be the set of sequences with exactly k descents and let $h_k = |H_k|$, the number of such sequences. Table 1.1 shows that $h_0 = h_3$ and $h_1 = h_2$, that is, *the "h-vector"* (h_0, h_1, h_2, h_3) *is symmetric.* MacMahon proved in general that

$$h_k = h_{n-1-r-k} \quad (0 \leqslant k \leqslant n - 1 - r).$$

MacMahon's proof used generating functions: he did not directly establish a one-to-one correspondence between H_k and $H_{n-1-r-k}$. Indeed, writes Knuth, "No very simple correspondence is evident" except in trivial cases. (Knuth then goes on to establish such a bijection—an algorithm, really—using Foata's idea of expressing multipermutations as products of cycles [9, pp. 24–29, 43–44].)

A curious result, to be sure—"quite surprising," Knuth says—but does it tell us anything about anything else? That is, does it *generalize*?

Generalize *how*? one might ask. To answer that question, we must translate MacMahon's result into the language of ordered sets.

The plan of this paper is as follows. All definitions are contained in Section 5. In Section 2 we reveal Stanley's generalization of MacMahon's theorem. In Section 3 we state Stanley's problem. In Section 4 we mention related results from the literature. In Section 5 we solve Stanley's problem. In Section 6 we illustrate our solution with an example. In Section 7 we describe avenues for further research. In the appendix we illustrate posets described in the main body of this work. In Section 1 we give a plan of the paper...

2. Everything I needed to know I learned from the four-element posets

Instead of multipermutations of words with the letters

$$1, \ldots, 1, 2, \ldots, 2, 3, \ldots, 3, \ldots, m, \ldots, m,$$

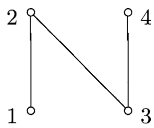
let us use *permutations* of the set $1, 2, \ldots, n$. The translation is illustrated in Fig. 2.1.

Fig. 2.1. Translating MacMahon's theorem to the language of posets.

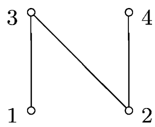
In any shuffling, such as 211212, replace the first 1 by 1, replace the second 1 by 2, replace the third 1 by 3, ..., replace the $(r+1)$st 1 by $r+1$; replace the first 2 by $r+2$, etc.; thus 211212 becomes 412536. Of course, we cannot get *any* permutation on n letters this way; we only get a permutation if, whenever $\rho < \rho'$ (ρ, ρ' elements of the poset on the right of Fig. 2.1), the numerical label of ρ appears to the left of the label for ρ'. Such a permutation is called a *linear extension* of the poset. (It is clear that a shuffling has k descents if and only if its translate does.)

A labelling of the elements of a finite poset with the letters $1, \ldots, n$ so that $123 \cdots n$ is a linear extension is called a *natural labelling*. Given a finite poset P with a natural labelling, we can define H_k to be the set of linear extensions (permutations compatible with the order on P) with k descents, and set $h_k = |H_k|$ as before.

Figure 2.2 shows a four-element poset with an unnatural labelling (illegal in some states); that same poset with *two* different natural labellings; and their corresponding sets of

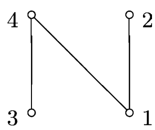


(a) The poset N with an unnatural labelling.



1234	0 descents
2134	1 descent
1243	1 descent
2413	1 descent
2143	2 descents

(b) The naturally labelled poset N has $h_0 = 1$, $h_1 = 3$, $h_2 = 1$.



1234	0 descents
1324	1 descent
1342	1 descent
3124	1 descent
3142	2 descents

(c) The poset N with another natural labelling.

Fig. 2.2.

linear extensions. Note that, while the *set* H_k depends on the natural labelling, the number h_k does not. (See, for instance, [20, Theorem 3.12.1].)

Figure A.1 in the appendix lists some other posets along with their h-vectors $(h_0, h_1, h_2, \ldots)$. (Strictly speaking, these are the h-vectors of the order complexes of the lattice of down-sets of these posets; see [1, Section 5.1] and [23, Section 8.3].)

Note that when P is an antichain, we get the classical *eulerian numbers*, and also note that a standard Young tableau is just a linear extension of a certain poset [14,16, pp. 43–44].

To illustrate, in Table A.1 we list all 24 permutations on four letters (Fig. A.1(f)), and mark those that are not linear extensions of the naturally labelled poset of Fig. A.1(g). Table A.2 lists the linear extensions of the naturally labelled poset of Fig. A.1(k); Table A.3 the linear extensions of the posets of Figs. 2.1 and A.1(l); and Table A.4 the linear extensions of the poset of Fig. A.1(m).

We note that, for each poset P, the index of the largest non-zero h_k is $k = n - 1 - r$, where $n = |P|$ and $r + 1$ is the cardinality of the longest chain (totally ordered subset). (See the easy Lemma 5.1 or [16, Theorem 16.1].) Moreover, the h-vector $(h_0, \ldots, h_{n-1-r})$ is symmetric just when P is *ranked* (*graded*), that is, when every maximal chain has the same cardinality. This is the content of Stanley's generalization of MacMahon's theorem.

Theorem (Stanley). *Let P be a finite naturally labelled poset. Let $\mathcal{L}(P)$ be the set of linear extensions of P, and, for every $\pi \in \mathcal{L}(P)$, let $d(\pi)$ be the number of descents of π. Let M be* $\max\{d(\pi) \mid \pi \in \mathcal{L}(P)\}$*. Then the following are equivalent*:

(i) $h_k = h_{M-k}$ *for* $0 \leqslant k \leqslant M$,
(ii) P *is ranked.*

3. The statement of Stanley's problem from the 1981 Banff Conference on Ordered Sets

At the 1981 Banff Conference on Ordered Sets [13, p. 807], Stanley said, "About ten years ago I proved (the above result)." He went on to pose the following

Problem (Stanley, 1981). Find a combinatorial proof of this theorem. More precisely, when (ii) holds describe explicitly a bijection $f : \mathcal{L}(P) \xrightarrow{\cong} \mathcal{L}(P)$ such that $d(\pi) = M - d(f(\pi))$ for all $\pi \in \mathcal{L}(P)$.

(Stanley added, "It would even be interesting to do this for the case $P \cong \mathbf{r} \times \mathbf{s}$ (the product of an r-element chain and an s-element chain).")

We solve Stanley's problem by constructing a bijection

$$\Phi_{k,k} : H_k \to H_{n-1-r-k}$$

for $k \in \{0, \ldots, M = n - 1 - r\}$ where $|P| = n$ and every maximal chain has $r + 1$ elements (Theorem 5.8).

4. Background and previous results

Basic references on posets are [2] and [20, Chapter 3]. We will not assume a poset is ranked without explicitly saying so. Because of the vast literature on f-vectors and h-vectors of polytopes and posets, permutation statistics, etc., we limit ourselves to recalling results most directly related to the present work, results concerning inequalities for h-vectors (which are also called w-vectors). Relevant papers (albeit not essential for understanding this work) include the very interesting [15], as well as [6,7] (see its Corollary 2.6) and [8] (see its Theorem 2.4), where Hibi shows, invoking a commutative algebra result [19, Theorem 2.1], that

$$h_0 + h_1 + \cdots + h_k \leqslant h_M + h_{M-1} + \cdots + h_{M-k} \quad \left(0 \leqslant k \leqslant \left\lfloor \frac{M}{2} \right\rfloor\right).$$

He states the following

Conjecture (Hibi, 1991). *For* $0 \leqslant k \leqslant \lfloor \frac{M}{2} \rfloor$,

$$h_k \leqslant h_{M-k} \quad \textit{and} \quad h_0 \leqslant h_1 \leqslant \cdots \leqslant h_{\lfloor \frac{M}{2} \rfloor}.$$

In the proof of [5, Theorem 1.2], Gasharov provides a bijection from H_k to $H_{n-1-r-k}$ when the rank r of the poset is 1 or 2, where we use the definition of rank that says that an antichain has rank 0. (He also proves that the h-vector is unimodal.) He writes, “The proof that we provide for Theorem 1.2 can be considered combinatorial, although we do not explicitly exhibit the necessary injections as this would be rather cumbersome.”

Reiner and Welker [12] prove that, when P is ranked, the h-vector is symmetric and unimodal by invoking the (decidedly non-trivial) g-Theorem for simplicial polytopes [18]; but this is not a combinatorial proof.

Fix a poset P of cardinality n. Let $\Omega(P,m)$ denote the number of order-preserving maps from P to an m-element chain and let $\overline{\Omega}(P,m)$ denote the number of *strictly* order-preserving maps. These are polynomials in m (the *order polynomial* and the *strict order polynomial*, respectively). Stanley’s reciprocity theorem for order polynomials ([17, Proposition 2.1], [20, Corollary 4.5.15]) states that

$$\overline{\Omega}(P,m) = (-1)^n \Omega(P,-m).$$

(Kreweras concedes being initially unaware of Stanley’s results, but his exposition is still interesting [10].) Though partially hidden, our Proposition 5.5 really amounts to analyzing the reciprocity theorem and its ingredients from Stanley’s theory of P-partitions and considerations like those in [16, Section 18]. (We obtain the final bijection using the involution principle.)

Thus we see that Stanley could have solved Stanley’s problem by reading Stanley.

5. The solution to Stanley's problem

We will use the following notation and definitions throughout this section and the next.

All numbers will be non-negative integers. For $n \geqslant 0$, let $[n] := \{1, \dots, n\}$ and let $[n]_0 := \{0, \dots, n\}$. (If we have an expression like $\{1, \dots, n\}$ where $n = 0$, then we mean the empty set.) Let $|S|$ denote the cardinality of the finite set S. If X, Y, X', and Y' are sets with $X \cap Y = \emptyset$, and if $f : X \to X'$ and $g : Y \to Y'$ are functions, define $f \cup g : X \cup Y \to X' \cup Y'$ to be the function such that, for every $z \in X \cup Y$,

$$(f \cup g)(z) = \begin{cases} f(z) & \text{if } z \in X, \\ g(z) & \text{if } z \in Y. \end{cases}$$

A *multiset* is a family with repetitions (so $\{1, 2, 2, 3\} \neq \{1, 2, 3\}$ as multisets). We define cardinality, union, and complementation for multisets appropriately, so

$$\bigl|\{1, 2, 2, 3\}\bigr| = 4, \quad \{1, 2\} \cup \{2, 3\} = \{1, 2, 2, 3\}, \quad \text{and}$$

$$\{1, 2, 2, 3\} \setminus \{1, 2\} = \{2, 3\}.$$

For $k \geqslant 0$, let $\left(\!\binom{S}{k}\!\right)$ denote the family of cardinality k multisets with elements drawn from the set S; if $d_1, \dots, d_k$ are numbers ($k \geqslant 0$), the notation $\{d_1, \dots, d_k\}_{\leqslant}$ for the corresponding multiset indicates that

$$d_1 \leqslant \cdots \leqslant d_k.$$

Let P be a finite alphabet (set). If w is a word $\sigma_1 \cdots \sigma_k$ with k letters ($k \geqslant 0$; $\sigma_1, \dots, \sigma_k \in P$), the *length* $|w|$ of w is k; we say the letter σ_i *appears* in w ($i \in [k]$); and, if $1 \leqslant i < j \leqslant k$, that σ_i *appears to the left* of σ_j in w. If $w_1 = \sigma_1 \cdots \sigma_k$ and $w_2 = \tau_1 \cdots \tau_l$ are words ($k, l \geqslant 0$; $\sigma_1, \dots, \sigma_k, \tau_1, \dots, \tau_l \in P$), then the *concatenation* of w_1 and w_2, denoted $w_1 w_2$, is the word $\sigma_1 \cdots \sigma_k \tau_1 \cdots \tau_l$.

A non-empty finite poset P is *ranked* of *rank* r if all maximal chains (totally ordered subsets maximal with respect to set-inclusion) have the same cardinality $r + 1$; the *rank* $r(\rho)$ of an element $\rho \in P$ is the rank of the subposet $\{\rho' \in P \mid \rho' \leqslant \rho\}$.

Fix a finite ranked poset P of cardinality $n \geqslant 2$ and rank r. Fix an order-preserving bijection from P to the chain $[n]$ and label the elements of P as $\rho_1, \rho_2, \dots, \rho_n$ so that $\rho_i \mapsto i$ ($i \in [n]$). (This is called a *natural labelling*.)

If

$$w = \rho_{i_1} \cdots \rho_{i_k}$$

is a word drawn from the alphabet P (where $k \geqslant 0$; $i_1, \dots, i_k \in [n]$), then we say w is *in increasing order* if $i_1 < \cdots < i_k$; and *in decreasing order* if $i_1 > \cdots > i_k$.

A *linear extension* of P is a word

$$w = \rho_{i_1} \cdots \rho_{i_n} \quad \bigl(i_1, \dots, i_n \in [n]\bigr)$$

with n distinct letters such that, if $\rho < \rho'$ in P, then ρ appears to the left of ρ' in w. The *descent set* $D(w)$ of such a linear extension w is the set $\{j \in [n-1] \mid i_j > i_{j+1}\}$ and the *ascent set* $A(w)$ is $\{j \in [n-1] \mid i_j < i_{j+1}\}$; we say w has k *descents* and l *ascents* if $k = |D(w)|$ and $l = |A(w)|$. Let H_k be the set of linear extensions of P with k descents.

For the next three paragraphs, fix $k \in [n-1-r]_0$. If $l \leqslant k$, let

$$\mathcal{D}_{k,l} := \left\{ (w, D) \in H_l \times \left(\left({[n-1] \atop k} \right) \right) \,\middle|\, D(w) \subseteq D \right\}$$

and

$$\mathcal{A}_{k,l} := \left\{ (v, A) \in H_{n-1-r-l} \times \left(\left({[n-1] \atop r+k} \right) \right) \,\middle|\, A(v) \subseteq A \right\}.$$

Let $\mathcal{D}_k := \bigcup_{l=0}^{k} \mathcal{D}_{k,l}$ and $\mathcal{A}_k := \bigcup_{l=0}^{k} \mathcal{A}_{k,l}$.

For $(w, D) \in \mathcal{D}_k$, where $D = \{d_1, \ldots, d_k\}_{\leqslant}$, let the *canonical factorization* of w be

$$w = w_0 \cdots w_k$$

where, for each $i \in [k]$,

$$d_i = |w_0 \cdots w_{i-1}|.$$

For $\rho \in P$, define $o(\rho)$ to be the number $i \in [k]_0$ such that ρ appears in w_i.

For $(v, A) \in \mathcal{A}_k$, where $A = \{a_1, \ldots, a_{r+k}\}_{\leqslant}$, let the *canonical factorization* of v be

$$v = v_0 \cdots v_{r+k}$$

where, for each $j \in [r+k]$,

$$a_j = |v_0 \cdots v_{j-1}|.$$

For $\rho \in P$, define $q(\rho)$ to be the number $j \in [r+k]_0$ such that ρ appears in v_j.

Lemma 5.1. *For $l \in \{n-r, \ldots, n-1\}$, $H_l = \emptyset$.*

Proof. There is a maximal chain

$$\rho_{i_0} < \cdots < \rho_{i_r}$$

where $i_0, \ldots, i_r \in [n]$. Then $i_0 < \cdots < i_r$ so any linear extension of P contains at least r ascents. $\square$

Lemma 5.2. *Let $k \in [n-1-r]_0$.*

(1) *Let* $(w, D) \in \mathcal{D}_k$ *and let* $w = w_0 \cdots w_k$ *be the canonical factorization. Then, for* $i \in [k]_0$, w_i *is in increasing order.*
(2) *Let* $(v, A) \in \mathcal{A}_k$ *and let* $v = v_0 \cdots v_{r+k}$ *be the canonical factorization. Then, for* $j \in [r+k]_0$, v_j *is in decreasing order.*

Proof. (1) This follows from the fact that $D(w) \subseteq D$. (2) This follows from the fact that $A(v) \subseteq A$. □

Lemma 5.3. *Let* $k \in [n-1-r]_0$ *and suppose* $(v, A) \in \mathcal{A}_k$. *Then for* $\rho, \rho' \in P$ *such that* $\rho \leqslant \rho'$, *we have*

$$q(\rho) - r(\rho) \leqslant q(\rho') - r(\rho').$$

Proof. If $\rho < \rho'$, then $r(\rho) < r(\rho')$, so there is a saturated chain

$$\rho =: \rho_{i_{r(\rho)}} < \cdots < \rho_{i_{r(\rho')}} := \rho'$$

where $i_{r(\rho)}, \ldots, i_{r(\rho')} \in [n]$ with $i_{r(\rho)} < \cdots < i_{r(\rho')}$. By Lemma 5.2(2),

$$q(\rho) = q(\rho_{i_{r(\rho)}}) < \cdots < q(\rho_{i_{r(\rho')}}) = q(\rho')$$

and hence $q(\rho') - q(\rho) \geqslant r(\rho') - r(\rho)$. □

Corollary 5.4. *Let* $k \in [n-1-r]_0$ *and suppose* $(v, A) \in \mathcal{A}_k$. *Then for all* $\rho \in P$, $0 \leqslant q(\rho) - r(\rho) \leqslant k$.

Proof. There exist $\rho', \rho'' \in P$ such that $\rho'' \leqslant \rho \leqslant \rho'$ and $r(\rho'') = 0$ and $r(\rho') = r$. By Lemma 5.3,

$$0 \leqslant q(\rho'') = q(\rho'') - r(\rho'') \leqslant q(\rho) - r(\rho) \leqslant q(\rho') - r(\rho') \leqslant (r+k) - r = k. \qquad \square$$

Proposition 5.5. *Fix* $k \in [n-1-r]_0$.

Define a map $\phi_k : \mathcal{D}_k \to \mathcal{A}_k$ *in the following manner. Given* $(w, D) \in \mathcal{D}_k$, *define a sequence of words* $v_0, \ldots, v_{r+k}$ *by letting* $\rho \in P$ *appear in the word* $v_{o(\rho)+r(\rho)}$ *and writing each word in decreasing order. Let* $v = v_0 \cdots v_{r+k}$, *and, for each* $j \in [r+k]$, *let*

$$a_j := |v_0 \cdots v_{j-1}|$$

and let $A = \{a_1, \ldots, a_{r+k}\}$. *Set* $\phi_k(w, D) = (v, A)$.

Define a map $\psi_k : \mathcal{A}_k \to \mathcal{D}_k$ *in the following manner. Given* $(v, A) \in \mathcal{A}_k$, *define a sequence of words* $w_0, \ldots, w_k$ *by letting* $\rho \in P$ *appear in the word* $w_{q(\rho)-r(\rho)}$ *and writing each word in increasing order. Let* $w = w_0 \cdots w_k$, *and, for each* $i \in [k]$, *let*

$$d_i := |w_0 \cdots w_{i-1}|$$

and let $D = \{d_1, \ldots, d_k\}$. *Set* $\psi_k(v, A) = (w, D)$.

Then ϕ_k *and* ψ_k *are well-defined, mutually-inverse bijections.*

We illustrate this bijection in Section 6, which the reader might wish to read while going through the proof below.

Proof. In the first part of the proof, we show that ϕ_k is well defined. Select $(w, D) \in \mathcal{D}_k$. Let $w = w_0 \cdots w_k$ be the canonical factorization and let $v = v_0 \cdots v_{r+k}$ be as in the statement of the proposition. As $0 \leqslant o(\rho) + r(\rho) \leqslant k + r$ for each $\rho \in P$, v contains each letter of P exactly once.

We show that v is a linear extension. Let $\rho, \rho' \in P$ be such that $\rho < \rho'$. Then $r(\rho) < r(\rho')$ and $o(\rho) \leqslant o(\rho')$ (since w is a linear extension), so $o(\rho) + r(\rho) < o(\rho') + r(\rho')$. Thus ρ appears to the left of ρ' in v.

By Lemma 5.1, $|A(v)| \geqslant r$; and clearly $|A(v)| \leqslant r + k$ since v_j is in decreasing order for each $j \in [r + k]_0$. Letting $l := |A(v)| - r$ we see that $v \in H_{n-1-r-l}$.

Because $D \subseteq [n - 1]$ and $n \geqslant 1$, we know $|w_0|, |w_k| \geqslant 1$. The first letter in w_0 must have rank 0 and so will be in v_0; the last letter in w_k must have rank r and so will be in v_{r+k}. Hence $A \subseteq [n - 1]$. Because each of $v_0, \ldots, v_{r+k}$ is in decreasing order, $A(v) \subseteq A$. Hence $(v, A) \in \mathcal{A}_{k,l}$. Note that $v_0 \cdots v_{r+k}$ is the canonical factorization of v.

In the second part of the proof, we show that ψ_k is well defined. Select $(v, A) \in \mathcal{A}_k$. Let $v = v_0 \cdots v_{r+k}$ be the canonical factorization and let $w = w_0 \cdots w_k$ be as in the statement of the proposition. These words are well defined by Corollary 5.4; w contains each letter of P exactly once.

We show that w is a linear extension. Let $\rho, \rho' \in P$ be such that $\rho < \rho'$. By Lemma 5.3 and the fact that $w_0, \ldots, w_k$ are in increasing order, ρ appears to the left of ρ' in w.

The fact that $w_0, \ldots, w_k$ are in increasing order also says that $D(w) \subseteq D$. Because $A \subseteq [n - 1]$ and $n \geqslant 1$, we know $|v_0|, |v_{r+k}| \geqslant 1$. The first letter of v_0 must have rank 0 and so will be in w_0; the last letter of v_{r+k} must have rank r and so will be in w_k. Hence $D \subseteq [n - 1]$ and thus $(w, D) \in \mathcal{D}_k$. Note that $w_0 \cdots w_k$ is the canonical factorization of w.

Now again select $(w, D) \in \mathcal{D}_k$ and let $(v, A) = \phi_k(w, D)$ and $(w', D') = \psi_k(v, A)$. Let $w = w_0 \cdots w_k$, $w' = w'_0 \cdots w'_k$, and $v = v_0 \cdots v_{r+k}$ be the canonical factorizations of w, w', and v, respectively. For $i \in [k]_0$, $\rho \in P$ appears in w_i if and only if it appears in $v_{i+r(\rho)}$ if and only if it appears in w'_i; thus $w_i = w'_i$. Hence $w = w'$ and $D = D'$.

Select $(v, A) \in \mathcal{A}_k$ and let $(w, D) = \psi_k(v, A)$ and $(v', A') = \phi_k(w, D)$. Let $v = v_0 \cdots v_{r+k}$, $v' = v'_0 \cdots v'_{r+k}$, and $w = w_0 \cdots w_k$ be the canonical factorizations of v, v', and w, respectively. For $j \in [r + k]_0$, $\rho \in P$ appears in v_j if and only if it appears in $w_{j-r(\rho)}$ if and only if it appears in v'_j; thus $v_j = v'_j$. Hence $v = v'$ and $A = A'$. □

Lemma 5.6. *Let $k, l \in [n - 1 - r]_0$ where $l \leqslant k$. Suppose there exists a bijection $\Phi_{l,l} : \mathcal{D}_{l,l} \to \mathcal{A}_{l,l}$ with inverse $\Psi_{l,l} : \mathcal{A}_{l,l} \to \mathcal{D}_{l,l}$.*

Define a map

$$\Phi_{k,l} : \mathcal{D}_{k,l} \to \mathcal{A}_{k,l}$$

as follows: for all $(w, D) \in \mathcal{D}_{k,l}$, $\Phi_{k,l}(w, D) := (v, A)$ where

$$\big(v, A(v)\big) = \Phi_{l,l}\big(w, D(w)\big) \quad \textit{and} \quad A = A(v) \cup \big[D \setminus D(w)\big]$$

(*a union of multisets*).

Define a map

$$\Psi_{k,l}:\mathcal{A}_{k,l}\to\mathcal{D}_{k,l}$$

as follows: *for all* $(v,A)\in\mathcal{A}_{k,l}$, $\Psi_{k,l}(v,A)=(w,D)$ *where*

$$\big(w,D(w)\big)=\Psi_{l,l}\big(v,A(v)\big)\quad\textit{and}\quad D=D(w)\cup\big[A\setminus A(v)\big]$$

(*a union of multisets*).

Then $\Phi_{k,l}$ *and* $\Psi_{k,l}$ *are well-defined, mutually-inverse bijections.*

Proof. First we show that $\Phi_{k,l}$ is well defined. With $(w,D)\in\mathcal{D}_{k,l}$ as above, $|A|=|A(v)|+|D|-|D(w)|=r+l+k-l=r+k$.

Next, we show that $\Psi_{k,l}$ is well defined. With $(v,A)\in\mathcal{A}_{k,l}$ as above, $|D|=|D(w)|+|A|-|A(v)|=l+r+k-(r+l)=k$.

Now suppose

$$(w,D)\in\mathcal{D}_{k,l},\quad (v,A)=\Phi_{k,l}(w,D),\quad\text{and}\quad \big(w',D'\big)=\Psi_{k,l}(v,A).$$

Clearly $w=w'$ (because $\Phi_{l,l}$ and $\Psi_{l,l}$ are inverses). Also,

$$\begin{aligned}D'&=D(w)\cup\big[\big(A(v)\cup\big[D\setminus D(w)\big]\big)\setminus A(v)\big]\\&=D(w)\cup\big(D\setminus D(w)\big)=D\end{aligned}$$

since $D(w)\subseteq D$.

Finally, suppose

$$(v,A)\in\mathcal{A}_{k,l},\quad (w,D)=\Psi_{k,l}(v,A),\quad\text{and}\quad \big(v',A'\big)=\Phi_{k,l}(w,D).$$

Clearly $v=v'$. Also,

$$\begin{aligned}A'&=A(v)\cup\big[\big(D(w)\cup\big[A\setminus A(v)\big]\big)\setminus D(w)\big]\\&=A(v)\cup\big(A\setminus A(v)\big)=A\end{aligned}$$

since $A(v)\subseteq A$. □

Lemma 5.7 (Involution Principle, q.v. [4,20, §2.6]). *Let* X, Y, X', *and* Y' *be finite sets with* $X\cap Y=\emptyset=X'\cap Y'$. *Let* $\Phi_X:X\to X'$ *and* $\phi:X\cup Y\to X'\cup Y'$ *be bijections with inverses* $\Psi_X:X'\to X$ *and* $\psi:X'\cup Y'\to X\cup Y$, *respectively.*

Define a map $\Phi_Y:Y\to Y'$ *as follows. For all* $y\in Y$, *let* $t\geqslant 0$ *be the smallest non-negative integer such that*

$$\big((\phi\circ\Psi_X)^t\circ\phi\big)(y)=:y'\in Y'$$

(*such a* t *must exist*) *and let* $\Phi_Y(y):=y'$.

Define a map $\Psi_Y : Y' \to Y$ as follows. For all $y' \in Y'$, let $t \geqslant 0$ be the smallest non-negative integer such that

$$\big((\psi \circ \Phi_X)^t \circ \psi\big)\big(y'\big) =: y \in Y$$

and let $\Psi_Y(y') := y$.

Then Φ_Y and Ψ_Y are well-defined, mutually-inverse bijections.

Theorem 5.8. *Let P be a finite ranked poset of cardinality $n \geqslant 2$ and rank r. Let $k \in [n-1-r]_0$.*

Construct an explicit bijection $\Phi_{k,k} : H_k \to H_{n-1-r-k}$ in the following manner. (*We identify H_l with $\mathcal{D}_{l,l}$ and $H_{n-1-r-l}$ with $\mathcal{A}_{l,l}$ for all $l \leqslant k$.*)

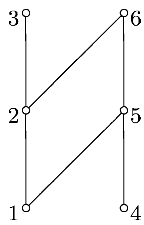
If $k = 0$, use the map ϕ_0 of Proposition 5.5.

If $k \geqslant 1$, first construct the bijections $\Phi_{l,l} : H_l \to H_{n-1-r-l}$ for $l \in [k-1]_0$; then construct the bijections $\Phi_{k,l} : \mathcal{D}_{k,l} \to \mathcal{A}_{k,l}$ as per Lemma 5.6. *Use the involution principle of Lemma* 5.7 *with $X = \bigcup_{l=0}^{k-1} \mathcal{D}_{k,l}$, $Y = H_k$, $X' = \bigcup_{l=0}^{k-1} \mathcal{A}_{k,l}$, $Y' = H_{n-1-r-k}$, $\Phi_X = \bigcup_{l=0}^{k-1} \Phi_{k,l}$, and $\phi = \phi_k$* (*the map of Proposition* 5.5).

Thus we solve the problem of Stanley from the 1981 Banff Conference on Ordered Sets.

6. An example of the bijection solving Stanley's problem

Consider the ranked poset of Fig. 6.1 with $n = 6$ and $r = 2$. Its h-vector is $(1, 6, 6, 1)$; see Table 6.1 for all of its linear extensions.



$h_0 = 1$ $h_1 = 6$ $h_2 = 6$ $h_3 = 1$

$n = 6$ $r = 2$

Fig. 6.1. A poset used to illustrate Theorem 5.8.

Table 6.1
Linear extensions of the poset of Fig. 6.1

H_0	H_1	H_2	H_3
123456	124356	415236	415263
	124536	412563	
	124563	412536	
	142356	145263	
	145236	142563	
	412356	142536	

For $k=0$ we have

$$\mathcal{D}_{0,0}=\{(123456,\emptyset)\} \quad \text{and} \quad \mathcal{A}_{0,0}=\{(41\ 52\ 63,24)\}.$$

(For clarity, we leave out the braces and commas when listing the multisets.) We leave it to the reader to guess the map ϕ_0 of Proposition 5.5 (and hence the map $\Phi_{0,0}$ of Theorem 5.8).

For $k=1$, we have

$$\mathcal{D}_{1,0}=\{(1\ 23456,1),(12\ 3456,2),(123\ 456,3),(1234\ 56,4),(12345\ 6,5)\}$$

and

$$\mathcal{D}_{1,1}=\{(124\ 356,3),(1245\ 36,4),(12456\ 3,5),\\ (14\ 2356,2),(145\ 236,3),(4\ 12356,1)\}.$$

We also have

$$\mathcal{A}_{1,0}=\{(4\ 1\ 52\ 63,124),(41\ \ 52\ 63,224),(41\ 5\ 2\ 63,234),\\ (41\ 52\ \ 63,244),(41\ 52\ 6\ 3,245)\}$$

and

$$\mathcal{A}_{1,1}=\{(41\ 52\ 3\ 6,245),(41\ 2\ 5\ 63,234),(41\ 2\ 53\ 6,235),\\ (1\ 4\ 52\ 63,124),(1\ 42\ 5\ 63,134),(1\ 42\ 53\ 6,135)\}.$$

We describe the map ϕ_1 of Proposition 5.5 by using spaces to delineate the factors in the canonical factorizations. (See Table 6.2.)

The map $\Phi_{1,0}$ of Lemma 5.6 is given by

Table 6.2
The map ϕ_1

D	w_0	w_1	$\xrightarrow{\phi_1}$	v_0	v_1	v_2	v_3	A
1	1	23456		1	4	52	63	124
2	12	3456		1	42	5	63	134
3	123	456		1	42	53	6	135
4	1234	56		41	2	53	6	235
5	12345	6		41	52	3	6	245
3	124	356		41	2	5	63	234
4	1245	36		41	52		63	244
5	12456	3		41	52	6	3	245
2	14	2356		41		52	63	224
3	145	236		41	5	2	63	234
1	4	12356		4	1	52	63	124

$$(1\ 23456, 1) \xrightarrow{\Phi_{1,0}} (4\ 1\ 52\ 63, 124)$$

$$(12\ 3456, 2) \xrightarrow{\Phi_{1,0}} (41\ \ 52\ 63, 224)$$

$$(123\ 456, 3) \xrightarrow{\Phi_{1,0}} (41\ 5\ 2\ 63, 234)$$

$$(1234\ 56, 4) \xrightarrow{\Phi_{1,0}} (41\ 52\ \ 63, 244)$$

$$(12345\ 6, 5) \xrightarrow{\Phi_{1,0}} (41\ 52\ 6\ 3, 245)$$

Finally, we can compute $\Phi_{1,1}$ using the involution principle:

$$\begin{array}{lclclcl}
124356 & \xrightarrow{\phi_1} & 412563 & & & & \\
124536 & \xrightarrow{\phi_1} & (41\ 52\ \ 63, 244) & \xrightarrow{\Psi_{1,0}} & (1234\ 56, 4) & \xrightarrow{\phi_1} & 412536 \\
124563 & \xrightarrow{\phi_1} & (41\ 52\ 6\ 3, 245) & \xrightarrow{\Psi_{1,0}} & (12345\ 6, 5) & \xrightarrow{\phi_1} & 415236 \\
142356 & \xrightarrow{\phi_1} & (41\ \ 52\ 63, 224) & \xrightarrow{\Psi_{1,0}} & (12\ 3456, 2) & \xrightarrow{\phi_1} & 142563 \\
145236 & \xrightarrow{\phi_1} & (41\ 5\ 2\ 63, 234) & \xrightarrow{\Psi_{1,0}} & (123\ 456, 3) & \xrightarrow{\phi_1} & 142536 \\
412356 & \xrightarrow{\phi_1} & (4\ 1\ 52\ 63, 124) & \xrightarrow{\Psi_{1,0}} & (1\ 23456, 1) & \xrightarrow{\phi_1} & 145263
\end{array}$$

Hence the bijection

$$\Phi_{1,1} : H_1 \to H_2$$

is given by

$$124356 \xrightarrow{\Phi_{1,1}} 412563$$

$$124536 \xrightarrow{\Phi_{1,1}} 412536$$

$$124563 \xrightarrow{\Phi_{1,1}} 415236$$

$$142356 \xrightarrow{\Phi_{1,1}} 142563$$

$$145236 \xrightarrow{\Phi_{1,1}} 142536$$

$$412356 \xrightarrow{\Phi_{1,1}} 145263$$

7. The future of an injection

While we have solved the problem of Stanley, our results could be improved in three ways. First, our bijection works for an arbitrary ranked poset with an arbitrary natural labelling, but there may be a more “natural” bijection for particular types of ranked posets with particular natural labellings. So it would still be satisfying to construct the bijection for a product of two chains. Second, the part of our bijection where we invoke the involution principle can probably be described even more explicitly in a manner reminiscent of jeu de taquin (although, needless to say, without the same far-reaching consequences).

Third, one could perhaps prove that $h_k \leqslant h_{n-1-r-k}$ for an arbitrary (not necessarily ranked) poset of cardinality n and height r ($k \leqslant \lfloor \frac{n-1-r}{2} \rfloor$) by refining our solution to Stanley's problem.

A. Poset menagerie

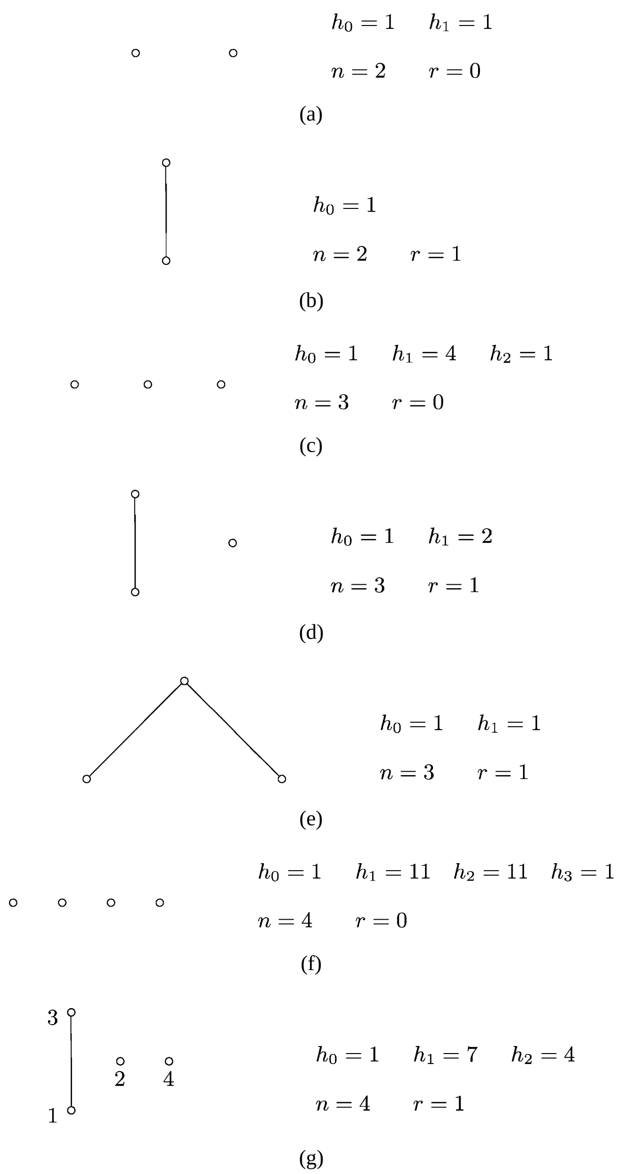
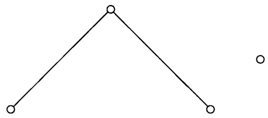


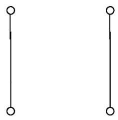
Fig. A.1. Examples of h-vectors.



$h_0 = 1$ $h_1 = 5$ $h_2 = 2$

$n = 4$ $r = 1$

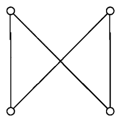
(h)



$h_0 = 1$ $h_1 = 4$ $h_2 = 1$

$n = 4$ $r = 1$

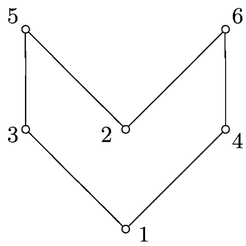
(i)



$h_0 = 1$ $h_1 = 2$ $h_2 = 1$

$n = 4$ $r = 1$

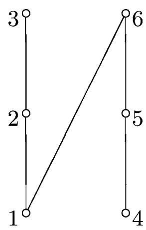
(j)



$h_0 = 1$ $h_1 = 8$ $h_2 = 11$ $h_3 = 2$

$n = 6$ $r = 2$

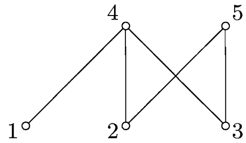
(k)



$h_0 = 1$ $h_1 = 8$ $h_2 = 9$ $h_3 = 1$

$n = 6$ $r = 2$

(l)

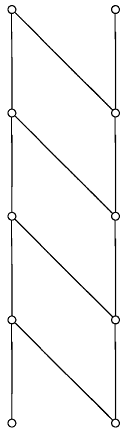


$h_0 = 1$ $h_1 = 6$ $h_2 = 6$ $h_3 = 1$

$n = 5$ $r = 1$

(m)

Fig. A.1. (Continued.)



$h_0 = 1 \quad h_1 = 15 \quad h_2 = 50$

$h_3 = 50 \quad h_4 = 15 \quad h_5 = 1$

$n = 10 \quad r = 4$

(n)

Fig. A.1. (Continued.)

Table A.1
Linear extensions of $\mathbf{1}+\mathbf{1}+\mathbf{1}+\mathbf{1}$ and $\mathbf{1}+\mathbf{1}+\mathbf{2}$ (the latter unmarked)

H_0	H_1	H_2	H_3
1234	2134	4312∗	4321∗
	3124∗	4213	
	4123	3214∗	
	1324	4231∗	
	1423	3241∗	
	2314∗	4132	
	2413	3142∗	
	3412∗	2143	
	1243	3421∗	
	1342	2431∗	
	2341∗	1432	

Table A.2
Linear extensions of the poset of Fig. A.1(k)

H_0	H_1	H_2	H_3
123456	123465	124365	143265
	123546	132465	214365
	124356	132546	
	124635	134265	
	132456	142365	
	134256	143256	
	142356	213465	
	213456	142635	
		213546	
		214356	
		214635	

Table A.3
Linear extensions of the posets of Figs. 2.1 and A.1(l) (the latter unmarked)

H_0	H_1	H_2	H_3
123456	124356	451623	415263
	124536	415623	
	124563	415236	
	142356	451263	
	145236	412563	
	145623	412536	
	412356	145263	
	451236	142563	
	456123∗	142536	

Table A.4
Linear extensions of the poset of Fig. A.1(m)

H_0	H_1	H_2	H_3
12345	13245	13254	32154
	21345	21354	
	12354	32145	
	31245	31254	
	23145	23154	
	23514	32514	

Acknowledgments

The author thanks H.A. Priestley for serving as his host at Oxford University while the author was a Fulbright Distinguished Scholar to the United Kingdom, and Richard Stanley for serving as his host at the Massachusetts Institute of Technology. The author thanks Amy Myers for extensive discussions on this problem [3]. The author thanks Mitch Harris for referring him to the ballot problem; Paul Edelman referred him to restricted growth functions (see, for instance, [21, p. 18] and [22]). The author also thanks André Kézdy and Dan Kleitman.

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