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Multifractal Models in Finance: Their Origin, Properties, and Applications

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1 Introduction

One of the most important tasks in financial economics is the modeling and forecasting of price fluctuations of risky assets. For analysts and policy makers volatility is a key variable for understanding market fluctuations. Analysts need accurate forecasts of volatility as an indispensable input for tasks such as risk management, portfolio allocation, value-at-risk assessment, and option and futures pricing. Asset market volatility also plays an important role in monetary policy. Repercussions from the recent financial crisis on the global economy show how important it is to take into account financial market volatility in conducting effective monetary policy.

In financial markets, volatility is a measure for fluctuations of the price p of a financial instrument over time. It cannot be directly observed, but has to be estimated via appropriate measures or as a component of a stochastic asset pricing model. As an ingredient of such a model, volatility may be a latent stochastic variable itself (as it is in so-called stochastic volatility models as well as in most multifractal models) or it might be a deterministic variable at any time t (as it is the case in so-called GARCH type models). For empirical data, volatility may simply be calculated as the sample variance or sample standard deviation. Ding *et al.* (1993) propose using absolute returns for estimating volatility. Davidian and Carroll (1987) demonstrate that this measure is more robust against asymmetry and non-normality than others (cf. also Taylor, (1986); Ederinton and Guan, (2005)). Another way to measure daily volatility is to use squared returns or any other absolute power of returns. Indeed, different powers show slightly different time-series characteristics, and the multifractal model is designed to capture the complete range of behavior of absolute moments.

Recently, the concept of realized volatility (RV) has been developed by Andersen *et al.* (2001*b*) as an alternative measure of the variability of asset prices (cf. also Bandorff-Nielsen and Shephard (2002*a*)). The notion of RV means that daily volatility is estimated by summing up intra-day squared returns. This approach is based on the theory of quadratic variation which suggests that RV should provide a consistent and highly efficient non-parametric estimator of asset return volatility over a given discrete interval under relatively parsimonious assumptions on the underlying data generating process. Other methods used for measuring volatility are: the maximum likelihood method developed by Ball and Torous (1984), or the high-low method proposed by Parkinson (1980). All these measures of financial market volatility show salient features which are well documented as *stylized facts*: volatility clustering, asymmetry and mean reversion, comovements of volatilities across assets and financial markets, stronger correlation of volatility compared to that of raw returns, (semi-) heavy-tails of the distribution of returns, anomalous scaling behavior, changes in shape of the return distribution over time horizons, leverage effects, asymmetric lead-lag correlation of volatilities, strong seasonality, and some dependence of scaling exponents on market structure, cf. sec. 2.

During the last decades, an immense body of theoretical and empirical studies has been devoted to formulate appropriate volatility models (cf. Andersen *et al.* (2006) for a recent review on volatility modeling and Poon and Granger (2003) for a review on volatility forecasting). With Mandelbrot's famous work on the fluctuations of cotton prices in the early sixties (cf. Mandelbrot, (1963)), economists had already learned that the standard Geometric Brownian motion proposed by Bachelier (1900) is unable to reproduce these stylized facts. In particular, the fat tails and the strong correlation observed in volatility are in sharp contrast to the "mild", uncorrelated fluctuations implied by models with Brownian random terms. A first step toward covering time-variation of volatility had been taken with models using mixtures of distributions as proposed by Clark (1973) and Kon (1984). Econometric modeling of asset price dynamics with time-varying volatility got started with the generalized autoregressive conditional heteroscedasticity (GARCH) family and it numerous extensions (cf. Engle, (1982)). The closely related class of stochastic volatility (SV) models adds randomness to the dynamic law governing the time variation of second moments (cf. Ghysels *et al.* (1996) and Shephard (1996) for a review on SV models and their applications).

In this chapter, the focus is on a new, alternative avenue for modeling and forecasting volatility developed in the literature over the last fifteen years or so. In contrast to the existing models the source of heterogeneity of volatility in these new models stems from the time-variation of local regularity in the price path (cf. Fisher *et al.* (1997)). The background of these models is the theory of multifractal measures that has originally been developed by Mandelbrot (1974) in order to model turbulent flows. These multifractal processes have initiated a broad current of literature in statistical physics refining and expanding the underlying concepts and models (cf. Kahane and Peyrière (1976), Holley and Waymire (1992), Falconer (1994), Arbeiter and Patzchke (1996), Barral (1999)). The formal analysis of such measures and processes, the so-called multifractal formalism, has been developed by Frisch and Parisi (1985), Mandelbrot (1989, 1990), and Evertz and Mandelbrot (1992), among others.

A number of early contributions have indeed pointed out certain similarities of volatility to fluid turbulence (cf. Vassilicos *et al.* (1994), Ghashghaie *et al.* (1996), Gallucio *et al.* (1997), Schmitt *et al.* (1999)), while theoretical modeling in finance using the concept of multifractality started with the adaptation to an asset-pricing framework of Mandelbrot's (1974) model by Mandelbrot *et al.* (1997).

Subsequent literature has moved from the more combinatorial style of the Multifractal Model of Assets Returns (MMAR) of Mandelbrot, Calvet and Fisher (developed in the sequence of Cowles Foundation working papers authored by Calvet *et al.* (1997), Fisher *et al.* (1997), and Mandelbrot *et al.* (1997)) to iterative, causal models of similar design principles: The Markov-Switching Multifractal (MSM) model proposed by Calvet and Fisher (2004) and the Multifractal Random Walk (MRW) by Bacry *et al.* (2001) constitute the second-generation of multifractal models that have more or less replaced the somewhat cumbersome (see below) first generation MMAR in empirical applications.

The rest of this chapter is organized as follows. Section 2 presents an overview over the salient stylized facts of financial data and discusses the potential of the classes of GARCH and stochastic volatility models to capture these stylized facts. In Section 3, we introduce the baseline concept of multifractal measures and processes and provide an overview over different specifications of multifractal volatility models. Section 4 introduces the different approaches to estimate MF models and to forecast future volatility. Section 5 reviews empirical results on the application and performance of MF models and Section 6 concludes.

2 Stylized Facts of Financial Data

With the availability of high-frequency time series for many financial markets from about the sixties, their statistical properties became a topic explored in a large strand of literature to which economists, statisticians and physicists have contributed. The two main universal features or "stylized facts" characterizing practically every series of interest at the high-end of the frequency spectrum (daily or intra-daily) are known under the catchwords "fat tails" and "volatility clustering". The use of multifractal models is motivated to some extent by both of these properties, but multifractality (or, as it is sometime also called, multi-scaling or multi-affinity) proper is a more subtle feature that gradually started to emerge as an additional stylized fact since the nineties. In the following we will provide a short review of the historical development of our knowledge and the quantification of all these features capturing in passing also some lesser known statistical properties typically found in financial returns. The data format of interest is thereby typically returns, i.e. relative price changes, $\tilde{r}_t = \frac{p_t - p_{t-1}}{p_{t-1}}$ which for high-frequency data are almost identical to log-price changes $r_t = \ln(p_t) - \ln(p_{t-1})$ with p_t the price at time t (e.g., at daily or higher frequency).

2.1 Fat tails

This property relates to the shape of the unconditional distribution of a time series of returns. Historically, the first "hypothesis" on the distribution of price changes has been formulated by Bachelier (1900) who in his PhD thesis titled "Théorie de la Spéculation" assumed them to follow a Normal distribution. As is well known, many applied areas of financial economics such as option pricing theory (Black and Scholes, 1973) and portfolio theory (Markowitz, 1959) have followed this assumption, at least in their initial stages. The justification for this assumption is provided by the law of large numbers: If price changes at the smallest unit of time are independently and identically distributed random numbers (maybe driven by the stochastic flow of new information) returns over longer intervals can be seen as the sum of a large number of such *i.i.d.* observations, and irrespective of the distribution of their summands should under some weak additional assumptions converge to the Normal distribution. While this seemed plausible and the resulting Gaussian distribution would also come very handy for many applied purposes, Mandelbrot (1963) was the first to demonstrate that empirical data are distinctly non-Gaussian exhibiting excess kurtosis and higher probability mass in the center and in their tails than the Normal distribution. As can be confirmed with any sufficiently long record of stock market, foreign exchange or other financial data, the Gaussian distribution can always be rejected with statistical significance beyond all usual boundaries, and the observed largest historical price changes would be so unlikely under the Normal law that one would have to wait for horizons beyond at least the history of stock markets to observe them occur with non-negligible probability.

Mandelbrot (1963) and Fama (1963), as a consequence, proposed the so-called Lévy stable laws as an alternative for capturing these fat tails. This was motivated by the fact that in a generalized version of the central limit law dispensing with the assumption of a finite second moment, sums of *i.i.d.* random variables converge to these more general distributions (with the Normal being a special case of the Lévy stable obtained in the borderline case of a finite second moment). The desirable stability property, therefore, indicates the choice of the Lévy stable which also has a shape that -in the standard case of infinite variance- is characterized by fat tails. In a sense, the Lévy stable model remained undisputed for about three decades (although many areas of financial economics would rather continue to use the Normal as their working model), and economists indeed contributed to the advancement of statistical techniques for estimating the parameters of the Lévy distributions (Fama and Roll, 1971; McCulloch, 1986). When physicists started to explore financial time series, the Lévy stable law was discovered again

(Mantegna, 1991) although new developments in empirical finance had already allowed to reject this meanwhile time-honored hypothesis.

These new insights were basically due to a different perspective: Rather than attempting to model the entire distribution, one let "speak the tails for themselves". The mathematical foundations for such an approach are provided by statistical extreme value theory (e.g., Reiss and Thomas, 1997). Its basic tenet is that the extremes and the tail regions of a sample of *i.i.d.* random variables converge in distribution to one of only three types of limiting laws. For tails, these are: exponential decay, power-law decay and the behavior of distributions with finite endpoint of their support. Fat tails are often used as a synonym for power-law tails, so that the highest realizations of returns would obey a law like $Prob(x_t < x) \sim 1 - x^{-\alpha}$ after appropriate normalization (i.e. after some transformation $x_t = ar_t + b$). The universe of fat-tailed distributions can, then, be indexed by their tail index α with $\alpha \in (0, \infty)$. Lévy stable distributions are characterized by tail indices α below 2 (2 characterizing the case of the Normal distribution). All other distributions with a tail index smaller than 2 would converge under summation to the Lévy stable with the same index while all distributions with an asymptotic tail behavior with $\alpha > 2$ would converge under aggregation to the Gaussian. This demarcates the range of relevance of the standard central limit law and its generalized version.

Jansen and de Vries (1991), Koedijk *et al.* (1990) and Lux (1996*a*) are examples of a literature that emerged over the nineties using semi-parametric methods of inference to estimate the tail index without assuming a particular shape of the entire distribution. The outcome of these and other studies is a tail index α in the range of 3 to 4 that now counts as a stylized fact (cf. Guillaume *et al.* (1997), Gopikrishnan *et al.* (1998)). Intra-daily data nicely confirm results obtained for daily records in that they provide estimates for the tail index that are in line with the former (Dacorogna *et al.* (2001), Lux (2001*b*)), and, therefore, confirm the expected stability of the tail behavior under time aggregation as predicted by extreme-value theory. The Lévy stable hypothesis, thus, can be rejected (confidence intervals of α typically exclude the possibility of $\alpha < 2$). This agrees with the evidence that the variance stabilizes with increasing sample size and does not explode. Falling into the domain of attraction of the Normal distributions, the overall shape of the return distribution would have to change, i.e. get closer to the Normal under time aggregation.[3] This is indeed the case, as has been demon-

[3]While, in fact, the tail behavior would remain qualitatively the same under time aggregation, the asymptotic power law would apply in a more and more remote tail region only, and would, therefore, become less and less visible for finite data samples under aggregation. There is, thus, both convergence towards the Normal distribution and stability of power-law behavior in the tail

strated by Teichmoeller (1971) and many later authors. Hence, the basic finding on the unconditional distribution is that it converges toward the Gaussian, but is distinctly different from it at the daily (and higher) frequencies. Fig. 1 illustrates the very homogeneous and distinctly both non-Gaussian and non-Levy nature of stock price fluctuations. The four major South-African stocks displayed in the figure could be replaced by almost any other time series of stock markets, foreign exchange markets and a variety of other financial markets. Estimating the tail index α by a linear regression in this log-log plot would lead to numbers very close to the celebrated "cubic law".

The particular non-Normal shape then also motivates the quest for the best non-stable characterization at intermediate levels of aggregation. From a huge literature that has tried mixtures of Normals (Kon (1984)) as well as a broad range of generalized distributions (Eberlein and Keller, 1995; Behr and Pötter, 2009; Fergussen and Platen 2006) it appears that the distribution of daily returns is quite close to a Student$-t$ with three degrees of freedom. However, while a tail index between 3 and 4 is typically found for stock and foreign exchange markets, some other markets are sometimes found to have fatter tails (e.g., Koedijk *et al.* (1992)) for black market exchange rates, and Matia *et al.* (2002) for commodities).

Figure 1 about here

2.2 Volatility clustering

The slow convergence to the Normal might be explained by dependency in the time series of returns. Indeed, while the limiting laws of extreme value theory would still apply for certain deviations from *i.i.d.* behavior, dependency could slow down convergence dramatically leading to a long regime of pre-asymptotic behavior. That returns are characterized by a particular type of dependency has also been well known for long time, and is mentioned, for instance, by Mandelbrot (1969). This dependency is most pronounced and in fact, plainly visible in absolute returns, squared returns, or any other measure of the extent of fluctuations (volatility), cf. Fig. 2. In all these measure there is long lasting, highly significant autocorrelation (cf. Ding *et al.* (1993)). With sufficiently long time series, significant autocorrelation can be found for time lags (of daily data) up to a few years. This positive feedback is described as volatility clustering or "turbulent (tranquil) periods being more likely to be followed by still turbulent (tranquil) periods than vice versa". Whether there is (additional) dependency in the raw

under aggregation. While the former governs the complete shape of the distribution, the latter applies further and further out in the tail only and would only be observed with a sufficiently large number of observations.

returns is subject to debate. Most studies do not find sufficient evidence for giving up the martingale hypothesis although a long-lasting but small effect might be hard to capture statistically. Ausloos *et al.* (1999) is an example of a study claiming to have identified such effects. Lo (1991) has proposed a rigorous statistical test for long term dependence that mostly does not indicate deviations from the null hypothesis of short memory for raw asset returns, but strongly significant evidence of long memory in squared or absolute returns. Similarly as for the classification of types of tail behavior, short memory comes along with exponential decay of the autocorrelation function while one speaks of long memory if the decay follows a power-law. Evidence for the later type of behavior has also accumulated over time. Documentation of hyperbolic decline in the autocorrelations of squared returns can be found in Dacorogna *et al.* (1993), Crato and de Lima (1994), Lux (1996*a*) and Mills (1997). Lobato and Savin (1998) first claimed that such long-range memory in volatility measures is a universal stylized fact of financial markets while Lobato and Velasco (2000) document similar long-range dependence in trading volume. Again, particular market designs might lead to exceptions from the typical power-law behavior. Gençay *et al.* (2001) as well as Ausloos and Ivanova (2000) report untypical behavior in the managed floating of European currencies during the times of the European Monetary System. Presumably due to leverage effects, stock markets also exhibit correlation between volatility and raw (i.e., signed) returns (cf. LeBaron, 1992), that is absent in foreign exchange dates.

Figure 2 about here

2.3 Benchmark Models: GARCH and Stochastic Volatility

In financial econometrics, volatility clustering has since the eighties spawned a voluminous literature on a new class of stochastic processes capturing the dependency of second moments in a phenomenological way. Engle (1982) first introduced the ARCH (autoregressive conditional heteroscedasticity model) which has been generalized to GARCH by Bollerslev (1986). It models returns as a mixture of Normals with the current variance being driven by a deterministic difference equation:

$$r_t = h_t \varepsilon_t \qquad \text{with} \quad \varepsilon_t \sim N(0,1) \tag{1}$$

and

$$h_t = \alpha_0 + \sum_{i=1}^{p} \alpha_i r_{t-i}^2 + \sum_{j=1}^{q} \beta_j h_{t-j}, \;\; \alpha_0 > 0, \alpha_i, \beta_j > 0 \tag{2}$$

Empirical applications usually find a parsimonious GARCH(1,1) model (i.e., $p = q = 1$) sufficient, and when estimated, the sum of the parameters $\alpha_1 + \beta_1$ turns out to be close to the non-stationary case (or, expressed differently, mostly only a constraint on the parameters prevents them for exceeding 1 in their sum which would lead to non-stationary behavior). Different extensions of GARCH were developed in the literature with the objective to better capture the stylized facts. Among them there are: the Exponential GARCH (EGARCH) model proposed by Nelson (1991) that accounts for asymmetric behavior of returns, the Threshold GARCH (TGARCH) model of Rabemananjara and Zakoian (1993) which takes into account the leverage effects, the regime switching GARCH (RS-GARCH) developed by Cai (1994), and the Integrated GARCH (IGARCH) introduced by Engle and Bollerslev (1986) that allows for capturing high persistence observed in returns time series. Itô diffusion or jump-diffusion processes can be obtained as a continuous time limit of discrete GARCH sequences (cf. Nelson (1990), Drost and Werker (1996)).

To capture stochastic shocks to the variance process, Taylor (1986) introduced the class of stochastic volatility models whose instantaneous variance is driven by:

$$\ln(h_t) = k + \varphi \ln(h_{t-1}) + \tau \xi_t, \qquad \xi_t \sim N(0,1). \tag{3}$$

This approach as well has been refined and extended in many ways. The SV process is more flexible than the GARCH model and provides more mixing because of the co-existence of shocks to volatility and return innovations (cf. Gavrishchaka and Ganguli (2003)). In terms of statistical properties, one important drawback of at least the baseline formalizations (1) to (3) is their implied exponential decay of the autocorrelations of measures of volatility which is in contrast to the very long autocorrelations mentioned before. Both the elementary GARCH and the baseline SV model are characterized by only short-term rather than long-term dependence.

To capture long memory, GARCH and SV models have been expanded by allowing for an infinite number of lagged volatility terms instead of the limited number of lags appearing in (2) and (3). To obtain a compact characterization of the long memory feature a fractional differencing operator has been used in both extensions leading to the fractionally integrated GARCH (FIGARCH) model of Baillie *et al.* (1996) and the long-memory stochastic volatility model of Breidt *et al.* (1998).[4] An interesting intermediate approach is the so-called heterogenous

[4] The "self-excited multifractal model" proposed by Filimonov and Sornette (2011) appears closer to this model rather than to models from the class of multifractal processes discussed below.

ARCH (HARCH) model of Dacorogna *et al.* (1998) that considers returns at different time aggregation levels as determinants of the dynamic law governing current volatility. Under this model, *eq.* (2) would have to be replaced by

$$h_t = c_0 + \sum_{j=1}^{n} c_j r^2_{t,t-\Delta t_j}, \tag{4}$$

where $r_{t,t-\Delta t_j} = \ln(p_t) - \ln(p_{t-\Delta t_j})$ are returns computed over different frequencies. The development of this model was motivated by the finding that volatility on fine time scales can be explained to a larger extend by coarse-grained volatility than vice versa (Müller *et al.* (1997)). Hence, the right-hand side covers local volatility at various lower frequencies than the time step of the underlying data ($\Delta t_j = 2, 3, \ldots$). As we will see in the following, multifractal models have a closely related structure but model the hierarchy of volatility components in a multiplicative rather than additive format.

2.4 A New Stylized Fact: Multifractality

Both the hyperbolic decay of the unconditional pdf as well as the similarly hyperbolic decay of the autocorrelations of many measures of volatility (squared, absolute returns) would fall into the category of scaling laws in the natural sciences. The identification of such universal scaling laws in an area like finance has spawned the interest of natural scientists to further explore the behavior of financial data and to develop models to explain these characteristics (cf. Mantegna and Stanley (1996)). From this line of research, multifractality, multi-scaling or anomalous scaling emerged gradually over the nineties as a more subtle characteristic of financial data that motivated the adaptation of known generating mechanisms for multifractal processes from the natural sciences in empirical finance.

To define multifractality or multiscaling, we start with the more basic concepts of fractality or scaling. The defining property of fractality is the invariance of some characteristic under appropriate self-affine transformations. The power-law functions characterizing the pdf of returns and autocorrelations of volatility measures are scale-invariant properties, i.e., this behavior is preserved over different scales under appropriate transformations.[5] In a most general way, some property

[5] e.g., from the limiting power law the cdf of a process with hyperbolically decaying tails obeys $Prob(x_i < x) \approx x^{-\alpha}$ and obviously for any multiple of x the same law applies: $Prob(x_i < cx) \approx (cx)^{-\alpha} = c^{-\alpha}x^{-\alpha}$.

of an object or a process needs to fulfill a law like

$$x(ct) = c^H x(t) \tag{5}$$

in order to be classified as scale-invariant, where t is an appropriate measurement of a scale (e.g., time or distance). Strict validity of (5) holds for many of the objects that have been investigated in fractal geometry (Mandelbrot (1982)). In the framework of stochastic processes, such laws could only hold in distribution. In this case, Mandelbrot *et al.* (1997) speak of self-affine processes. An example of a well-known class of processes obeying such a scale invariance principle is fractional Brownian motion for which $x(t)$ is a series of realizations and $0 < H < 1$ is the Hurst index that determines the degree of persistence ($H > 0.5$) or anti-persistence ($H < 0.5$) of the process, $H = 0.5$ corresponding to Wiener Brownian motion with uncorrelated Gaussian increments. Fig. 2 shows the scaling behavior of different powers of returns (raw, absolute and squared returns) of a financial index as determined by a popular method for the estimation of the Hurst coefficient, H. The law (5) also determines the dependency structure of the increments of a process obeying such scaling behavior as well as their higher moments which show hyperbolic decline of their autocorrelations with an exponent depending linearly on H. Such linear dependence is called uni-scaling or uni-fractality. It also carries over asymptotically to processes that use a fractional process as generator for the variance dynamics, e.g. the long memory stochastic volatility model of Breidt *et al.* (1998).[6]

Multifractality or anamalous scaling allows for a richer variation of the behavior of a process across different scales by only imposing the more general relationship:

$$x(ct) \overset{d}{=} M(c)x(t) \equiv c^{H(c)}x(t), \tag{6}$$

where the scaling factor $M(c)$ is a random function with possibly different shape for different scales and $\overset{d}{=}$ denotes equality in distribution. The last equality of *eq.* (6) illustrates that this variability of scaling laws could be translated into variability of the index H which now is not constant anymore. One might also note the multiplicative nature of transitions between different scales: One moves from one scale to another via multiplication with a random factor $M(c)$. We will see below that multifractal measures or processes are constructed exactly in this way which implies a combinatorial, noncausal nature of these processes.

Multi-scaling in empirical data is typically identified by differences in the scaling

[6]For the somewhat degenerate FIGARCH model, the complete asymptotics have not yet been established, cf. Jach and Kokoska (2010).

behavior of different (absolute) moments:

$$\mathbb{E}\left[|x(t,\Delta t)|^q\right] = c(q)\Delta t^{qH(q)+1} = c(q)\Delta t^{\tau(q)+1}, \tag{7}$$

with $x(t,\Delta t) = x(t) - x(t-\Delta t)$, and $c(q)$ and $\tau(q)$ being deterministic functions of the order of the moment q. A similar equation could be established for uni-scaling processes, e.g. fractional Brownian motion, yielding

$$\mathbb{E}\left[|x(t,\Delta t)|^q\right] = c^H \Delta t^{qH+1}. \tag{8}$$

Hence, in terms of the behavior of moments, multifractality (anomalous scaling) is distinguished by a non-linear (typically concave) shape from the linear scaling of uni-fractal, self-affine processes. The standard tool to diagnose multifractality is, then, inspection of the empirical scaling behavior of an ensemble of moments. Such non-linear scaling is illustrated in Fig. 3 for three selected stock indices and a stochastic process with multifractal properties (the Markov-switching multifractal model introduced below). The traditional approach in the physics literature consists in extracting $\tau(q)$ from a chain of linear log-log fits of the behavior of various moments q for a certain selection of time aggregation steps Δt. One, therefore, uses regressions to the temporal scaling of moments of powers q:

$$\ln \mathbb{E}\left[|x(t,\Delta t)|^q\right] = a_0 + a_1 \ln(\Delta t) \tag{9}$$

and constructs the empirical $\tau(q)$ curve (for a selection of discrete q) from the ensemble of estimated regression coefficients for all q. An alternative and perhaps even more widespread approach for identification of multifractality looks at the varying scaling coefficients $H(q)$ in *eq.* (7). While the unique coefficient H of *eq.* (8) is usually denoted the Hurst coefficient, the multiplicity of such coefficients in multifractal processes is denoted as Hölder exponents. While the unique H quantifies a global scaling property of the underlying process, the Hölder exponents can be viewed as local scaling rates that govern various patches of a time series leading to a characteristically heterogeneous (or intermittent) appearance of such series. An example is displayed in Fig. 5 (principles of construction being explained below). Focusing on the concept of Hölder exponents, multifractality then amounts to identification of the range of such exponents rather than a degenerate single H as for uni-fractal processes. The so-called spectrum of Hölder exponents (or multifractal spectrum) can be obtained by the Legendre transformation[7] of the scaling function $\tau(q)$. Define $\alpha = \frac{d\tau}{dq}$, the Legendre transform $f(\alpha)$

[7] The Legendre transformation is a mathematical operation that transforms a function of a

of the function $\tau(q)$ is given by

$$f(\alpha) = \arg\min_q [q\alpha - \tau(q)], \tag{10}$$

where α is the Hölder exponent (the established notation for the counterpart of the constant Hurst exponent, H) and $f(\alpha)$ the multifractal spectrum that describes the distribution of the Hölder exponents. The local Hölder exponent quantifies the local scaling properties (local divergence) of the process at a given point in time, in other words, it measures the local regularity of the price process. In traditional time series models, the distribution of Hölder exponents is degenerate converging to a single such exponent (unique Hurst exponent) while multifractal measures are characterized by a continuum of Hölder exponents whose distribution is given by the Legendre transform, *eq.* (10), for its particular scaling function $\tau(q)$. The characterization of a multifractal process or measure by a distribution of local Hölder exponents underlines its heterogeneous nature with alternating calm and turbulent phases.

Empirical studies allowing for such a heterogeneity of scaling relations typically identify "anomalous scaling" (curvature of the empirical scaling functions or non-singularity of the Hölder spectrum) for financial data as illustrated in Fig. 3. Historically, the first example of such an analysis is Müller *et al.* (1990) followed by more and more similar findings reported mostly in the emerging econophysics literature (due to the fact that the underlying concepts were well-known in physics from research on turbulent flows, but were completely alien to financial economists). Examples include Vassilicos *et al.* (1994), Mantegna and Stanley (1995), Ghashghaie *et al.* (1996), Fisher *et al.* (1997), Schmitt *et al.* (1999), Fillol (2003), among others. Ureche-Rangau and de Rorthays (2009) show that both volatility and volume of Chinese stocks appear to have multifractal properties, a finding one should probably be able to confirm for other markets as well given the established long-term dependence and high cross-correlation between both measures (cf. Lobato and Velsasco (2000), who among others, also report long-term dependence of volume data). While econometricians have not been looking at scaling functions and Hölder spectrums, the indication of multifractality in the mentioned studies has nevertheless some counterpart in the economics literature: The well-known finding of Ding *et al.* (1993) that (i) different powers of returns have different degrees of long-term dependence and that (ii) the intensity of long-term dependence varies non-monotonically with q (with a maximum obtained around $q \approx 1$) is consistent with concavity of scaling functions and

coordinate, $g(x)$, into a new function $h(y)$ whose argument is the derivative of $g(x)$ with respect to x, i.e., $y = \frac{dg}{dx}$.

provides evidence for "anomalous" behavior form a slightly different perspective.

Multifractality, thus, provides a generalization of the well established finding of long-term dependence of volatility: Different measures of volatility are characterized by different degrees of long-term dependence in a way that reflects the typical anomalous behavior of multifractal processes. Accepting such behavior as a new stylized fact, the natural next step would be to design processes that could capture this universal finding together with other well-established stylized facts of financial data. New models would be required because none of the existing ones would be consistent with this type of behavior: baseline GARCH and SV models have only exponential decay of the autocorrelations of absolute powers of returns (short-range dependence), while their long memory counterparts (LMSV, FIGARCH) are characterized by uni-fractal scaling.[8]

One caveat is, however, in order here: Whether the scaling function and Hölder spectrum analysis provide sufficient evidence for multifractal behavior, is to some extent subject to dispute. A number of papers show that scaling in higher moments can be easily obtained in a spurious way without any underlying anomalous diffusion behavior. Lux (2004) pointed out that a non-linear shape of the empirical $\tau(q)$ function is still obtained for financial data after randomization of their temporal structure, so that the $\tau(q)$ and $f(\alpha)$ estimators are rather unreliable diagnostic instruments for the presence of multifractal structure in volatility. Apparent scaling has also been illustrated by Barndorff-Nielson and Prause (2001) as a consequence of fat tails in the absence of true scaling. It is very likely that standard volatility models would also lead to apparent multi-scaling that could be hard to distinguish from "true" multifractality via the diagnostic tools mentioned above.[9] Formally, it will always be possible to design processes without a certain type of (multi-)scaling behavior that are locally so close to "true" (multi-)scaling that these deviations will never be detected with pertinent diagnostic tools and restricted availability of data (cf. LeBaron, 2001; Lux, 2001*a*).

On the other hand, one might follow Mandelbrot's frequently voiced methodological premise to model apparently generic features of data by similarly generic models rather than using "fixes" (Mandelbrot (1997*a*)). Introducing amendments to existing models (e.g., GARCH, SV) to adapt those to new stylized facts might lead to highly parameterized setups that lack robustness when applied to data from different markets, while simple generating mechanisms for multifractal be-

[8]For FIGARCH this is so far only indicated by simulations, but given that- as for LMSV-FIGARCH consists of a uni-fractal ARFIMA process plugged into the variance equation, it seems plausible that it also has uni-fractal asymptotics.

[9]There is also a sizeable literature on spurious generation of fat tails and long-term dependence, cf. Granger and Teräsvirta (1999) or Kearns and Pagan (1997).

havior are available that could, in principle, capture the whole spectrum of time series properties highlighted above in a more parsimonious way. In addition, if one wants to account for multi-scaling proper (rather than as a spurious property) no avenue is known so far for equipping GARCH- or SV-type models with this property in a generic way. Hence, adapting in an appropriate way some known generating mechanism for multifractal behavior appears the only avenue available so far to come up with models that generically possess such features, and jointly reproduce all stylized facts of asset returns. The next section recollects the major steps in the development of multifractal models for asset-pricing applications.

Figure 3 about here

3 Multifractal Measures and Processes

In the following, we first explain the construction of a simple multifractal measure and show how one can generalize it along various dimensions. We, then, move on to multifractal processes designed as models for financial returns.

3.1 Multifractal Measures

Multifractal measures have a long history in physics dating back to the early seventies when Mandelbrot proposed a probabilistic approach for the distribution of energy in turbulent dissipation (e.g., Mandelbrot (1974)). Building upon earlier models of energy dissipation by Kolmogorov (1941, 1962) and Obukhov (1962), Mandelbrot proposed that energy should dissipate in a cascading process on a multifractal set from long to short scales. In this original setting, the multifractal set results from operations performed on probability measures. The construction of a multifractal "cascade" starts by assigning uniform probability to a bounded interval (e.g., the unit interval $[0, 1]$). In a first step, this interval is split up into two subintervals receiving fractions m_0 and $1 - m_0$, respectively, of the total probability mass of unity of their mother interval. In the simplest case, both subintervals have the same length (i.e., 0.5), but other choices are possible as well. In the next step, the two subintervals of the first stage of the cascade are split up again into similar subintervals (of length 0.25 each in the simplest case) receiving again fractions m_0 and $1-m_0$ of the probability mass of their "mother" intervals (cf. Fig. 4). In principle, this procedure is repeated *ad infinitum*. With this recipe, a heterogeneous, fractal distribution of the overall probability mass results which even for the most elementary cases has a perplexing visual resemblance to time series of volatility in financial markets. This construction clearly reflects the underlying idea of dissipation of energy from the long scales (the mother intervals) to the finer scales that preserve the joint influence of all the previous hierarchical levels in the built-up of the "cascade".

Many variations of the above generating mechanism of a simple Binomial multifractal could be thought of: Instead of always assigning probability m_0 to the left-hand descendent, this assignment could as well be randomized. Furthermore, one could think of more than two subintervals to be generated in each step (leading to multinomial cascades) or of using random numbers for m_0 instead of the same constant value. A popular example of the later generalization is the Lognormal multifractal model which draws the mass assigned to new branches of the cascade from a Lognormal distribution (cf. Mandelbrot, 1974; 1990). Note that for the Binomial cascade the overall mass over the unit interval is exactly conserved at any preasymptotic stage as well as in the limit $k \to \infty$, while mass is preserved only in expectation under appropriately normalized Lognormal multipliers, or multipliers following any other continuous function. Another straightforward generalization consists in splitting each interval on level j into an integer number b of pieces of equal length at level $j+1$. The grid-free Poisson multifractal measure developed by Calvet and Fisher (2001) is obtained by allowing for randomness in the construction of intervals. In this setting, a bounded interval is split into separate pieces with different mass by determining a random sequence T_n of change points. Overall mass is then distributed via random multipliers across the elements of the partition defined by the T_n. A multifractal sequence of measures is generated by a geometric increase of the frequency of arrivals of change points at different levels j $(j = 1, \ldots, k)$ of the cascade. As in the grid-based multifractal measures, the mass within any interval after the completion of the cascade is given by the product of all k random multipliers within that segment.

Note that all the above recipes can be interpreted as implementations (or examples) of the general form (6) that defines multifractality from the scaling behavior across scales. The recursive construction principles are, themselves, directly responsible for the multifractal properties of the pertinent limiting measures. The resulting measures, thus, obey multifractal scaling analogous to *eq.* 7. Denoting by μ a measure defined on $[0,1]$, this amounts to[10] $\mathbb{E}[\mu(t, t+\Delta t)^q] \sim c(q)(\Delta t)^{\tau(q)+1}$. Exact proofs for the convergence properties of such grid bound cascades have been provided by Kahane and Peyrière (1976). The "multifractal formalism" that had been developed after Mandelbrot's pioneering contribution consisted in the generalization and analytical penetration of

[10] For example, for the simplest case of the Binomial cascade one gets $\tau(q) = -\ln \mathbb{E}[M^q] - 1$ with $M \in \{m_0, 1-m_0\}$ with probability 0.5.

various multifractal measures following the above principles of construction (cf. Tél, 1988; Evertsz and Mandelbrot, 1992; Riedi, 2002). Typical questions of interest are the determination of the scaling function $\tau(\alpha)$ and the Hölder spectrum $f(\alpha)$, as well as the existence of moments in the limit of a cascade with infinite progression.

Figure 4 about here

3.2 Multifractal Models

3.2.1 Univariate Continuous-Time Multifractal Models

3.2.1.1 The Multifractal Model of Asset Returns

Multifractal measures have been adapted to asset-price modeling by using them as a "stochastic clock" for transformation of chronological time into business (or intrinsic) time. Formally, such a time transformation can be represented by stochastic subordination, with the time change represented by a stochastic process, say $\theta(t)$ denoted the "subordinating process", and the asset price change, $r(t)$, being given by a subordinated process (e.g. Brownian motion) measured in transformed time, $\theta(t)$. In this way, the homogenous subordinated process might be modulated in a way to give rise to realistic time series characteristics such as volatility clustering. The idea of stochastic subordination has been introduced in financial economics by Mandelbrot and Taylor (1967). A well-known later application of this principle is Clark (1973) who had used trading volume as a subordinator (cf. Ané and Geman, 2000, for recent extensions of this approach).

Mandelbrot *et al.* (1997) seems to be the first paper that went beyond establishing phenomenological proximity of financial data to multifractal scaling. They proposed a model, termed the Multifractal Model of Asset Returns (MMAR), in which a multifractal measure as introduced in sec. 3.1 serves as a time transformation from chronological time to business time. While the original paper has not been published in a journal, a synopsis of this entry and two companion papers (Calvet *et al.*, 1997; Fisher *et al.*, 1997) has appeared as Calvet and Fisher (2002). Several other contributions by Mandelbrot (1997b, 1999, 2001a, b, c) contain graphical discussions of the construction of the time-transformed returns of the MMAR process and simulations of examples of the MMAR as a data generating process. Formally, the MMAR assumes that returns $r(t)$ follow a compound process:

$$r(t) = B_H[\theta(t)], \tag{11}$$

in which an incremental fractional Brownian motion with Hurst index H, $B_H[\cdot]$, is subordinate to the cumulative distribution function $\theta(t)$ of a multifractal measure constructed along the above lines. When investigating the properties of this process, the (unifractal) scaling of the fractional Brownian motion has to be distinguished from the scaling behavior of the multifractal measure. The behavior of the compound process is determined by both, but its multi-scaling in absolute moments remains in place even for $H = 0.5$, i.e. Wiener Brownian motion. Under the restriction $H = 0.5$, the Brownian motion part becomes uncorrelated Wiener Brownian motion and the MMAR shows the martingale property of most standard asset pricing models. This model shares essential regularities observed in financial time series including long tails and long memory in volatility which both originate from the multifractal measure $\theta(t)$ applied for the transition from chronological time to "business time". The heterogenous sequence of the multifractal measure, then, serves to contract or expand time and, therefore, also contracts or expands locally the homogeneous second moment of the subordinate Brownian motion.

As pointed out above, different powers of such a measure have different decay rates of their autocovariances. Mandelbrot *et al.* (1997) demonstrate that the scaling behavior of the multifractal time transformation carries over to returns from the compound process (11) which would obey a scaling function $\tau_r(q) = \tau_\theta(qH)$. Similarly, the shape of the spectrum carries over from the time transformation to returns in the compound process via a simple relationship: $f_r(\alpha) = f_\theta(\alpha/H)$. By writing $\theta(t) = \int_0^t d\theta(t)$, it becomes clear that the incremental multifractal random measure $d\theta(t)$ (which is the limit of $\mu[t, t + \Delta t]$ for $\Delta t \to 0$ and k (the number of hierarchical levels) $\to \infty$) can been considered as the instantaneous stochastic volatility. As a result, MMAR essentially applies the multifractal measure to capture the time-dependency and non-homogeneity of volatility. Mandelbrot *et al.* (1997) and Calvet and Fisher (2002) discuss estimation of the underlying parameters of the MMAR model via matching of the $f(\alpha)$ and $\tau(\alpha)$ functions, and show that the temporal behavior of various absolute moments of typical financial data squares well with the theoretical results for the multifractal model.

Any possible implementation of the underlying multifractal measure could be used for the time-transformation $\theta(t)$. All examples considered in their papers built upon a binary cascade in which the time interval of interest (in place of the unit interval in the abstract operations on a measure described in sec. 3.1) is split repeatedly into subintervals of equal length. The so obtained subintervals are assigned fractions of the probability mass of their mother interval drawn from different types of random distributions: Binomial, Lognormal, Poisson and Gamma distributions are discussed in Calvet and Fisher (2002) each of those leading to a particular $\tau(\alpha)$ and $f(\alpha)$ function (known from previous literature) and similar behavior of the compound process according to the relations detailed above. Lux

(2001*c*) applies an alternative estimation procedure minimizing a Chi-square criterion for the fit of the implied unconditional distribution of the MMAR to the empirical one, and reports that one can obtain surprisingly good approximations to the empirical shape in this way. However, Lux (2004) documents that $\tau(\alpha)$ and $f(\alpha)$ functions are not very reliable as criteria for determination of the parameters of the MMAR as even after randomization of the underlying data, one still gets indication of temporal scaling structure via non-linear $\tau(\alpha)$ and $f(\alpha)$ shapes. Poor performance of such estimators is also expected on the ground of the slow convergence of their variance as demonstrated by Ossiander and Waymire (2000). One might also point out in this respect, that both functions are capturing various moments of the data, so using them for determination of parameters amounts to some sort of moment matching. It is, however, not obvious that the choice of weight of different moments implied by these functions would be statistically efficient.

While MMAR has not been pursued further in subsequent literature, estimation of alternative multifractal models has made use of efficient moment estimators as well as other more standard statistical techniques. The main drawback of the MMAR is, that despite the attractiveness of its stochastic properties, its practical applicability suffers from the combinatorial nature of the subordinator $\theta(t)$ and its non-stationarity due to the restriction of this measure to a bounded interval. These limitations have been overcome by the analogous iterative time series models introduced by Calvet and Fisher (2001, 2004). Leövey and Lux (2012) have also recently proposed a re-interpretation of the MMAR in which an infinite succession of multifractal cascades overcomes the limitation to a bounded interval, and the resulting overall process could be viewed as a stationary one.

It is interesting to relate the grid-bound construction of the MMAR to the "classical" formalization of stochastic processes for turbulence. Building upon previous work by Kolmogorov (1962) and Obukhov (1962) on the phenomenology of turbulence, Castaing *et al.* (1990) has introduced the following approach to replicate the scaling characteristics of turbulent flows:

$$x_i = \exp(\varepsilon_i)\xi_i, \tag{12}$$

with ξ_i and ε_i both following a Normal distribution $\xi_i \sim N(0, \sigma^2)$ and $\varepsilon_i \sim N(\ln(\sigma_0), \lambda^2)$, and ξ_i and ε_i mutually independent. This approach has been applied to various fluctuating phenomena in the natural sciences such as hadron collision (Carius and Ingelman (1990)), solar wind (Sorriso-Valvo *et al.* (1999)), and human heartbeat (Kiyono *et al.* (2004), (2005)). Replacing the uniform ε_i by the sum of hierarchically organized components, the resulting structure would closely resemble that of the MMAR model. Models in this vein have been investigated in physics by Kiyono *et al.* (2007) and Kiyono (2009). Based on the approach exemplified in *eq.* (12), Ghasghaie *et al.* (1996) elaborate on the similarities between turbulence in physics and financial fluctuations, but do not take into account the possibility of multifractality of the data generating process.

3.2.1.2 The MMAR with Poisson Multifractal Time Transformation

Already in Calvet and Fisher (2001), a new type of multifractal model has been introduced that overcomes some of the limitations of the MMAR as proposed by Mandelbrot *et al.* (1997) while -initially- preserving the formal structure of a subordinated process. Instead of the grid-based binary splitting of the underlying interval (or, more generally, the splitting of each mother interval into the same number of subintervals), they assume that $\theta(t)$ is obtained in a grid-free way by determining a Poisson sequence of change points for the multipliers at each hierarchical level of the cascade. Multipliers themselves might again be drawn from a Binomial, Lognormal (the standard cases), or any other distribution with positive support. Change points are determined by renewal times with exponential densities. At each change point t_n^i a new draw $M_{t_n}^i$ of cascade level i occurs from the distribution of the multipliers that is standardized in a way to ensure conservation of overall mass $\mathbb{E}[M_{t_n}^i] = 1$. In order to achieve the hierarchical nature of the cascade, the different levels i are characterized by a geometric progression of the frequencies of arrival $b^i\lambda$. Hence, the change points t_n^i follow level-specific densities $f(t_n^{(i)}; \lambda, b) = b^i \lambda \exp(-b^i \lambda t_n^i)$, for $i = 1, ..., k$. Similar grid-free constructions for multifractal measures are considered in Cioczek-Georges and Mandelbrot (1995) and Barral and Mandelbrot (2002). In the limit $k \to \infty$ the Poisson multifractal exhibits typical anomalous scaling, which again carries over from the time transformation $\theta(t)$ to the subordinate process for asset returns, $B_H[\theta(t)]$ in the way demonstrated by Mandelbrot *et al.* (1997).

The importance of this variation of the original grid-bound MMAR is that it provides an avenue towards constructing multifractal models (or models arbitrarily close to "true" multifractals) in a way that allows better statistical tractability. In particular, in contrast to the grid-bound MMAR, the Poisson multifractal possesses a Markov structure. Since the $t_n^{(i)}$ follow an exponential distribution, the probability of arrivals at any instant t is independent from past history. As an immediate consequence, the initial restriction upon its construction to a bounded interval in time $[0, T]$ is not really necessary, as the process can be continued when reaching the border $t = T$ in the very same way by which realizations have been generated within the interval $[0, T]$ without any disruption of its stochastic structure. This is not the case for the grid-based approach where one could, in principle, append a new cascade after $t = T$ which, however, would be completely uncorrelated with the previous one. Lux (2013) shows that the Poisson MMAR can also be interpreted as a regime-switching diffusion process with 2^k different volatility states. This paper also derives the transient density of this process and shows how it could be utilized for exact maximum likelihood estimation of its parameters. Except for this contribution, the continuous-time Poisson multifractal has not been used itself in empirical applications, but it has motivated the development of the discrete Markov-switching multifractal model (MSM) that has become the most frequently applied version of multifractal models in empirical finance, cf. sec. 3.3.

3.2.1.3 Further Generalizations of Continuous-Time MMAR

In a foreword to the working paper version (2001) of their paper, Barral and Mandelbrot (2002) motivate the introduction of what they call "multifractal products of cylindrical pulses" by its greater flexibility compared to standard multifractals. They argue that this generalization should be useful in order to capture particularly the power-law behavior of financial returns. Again, in the construction of the cylindrical pulses the renewal times at different hierarchical levels are determined by Poisson processes whose intensities are not, however, connected via the geometric progression $b^i\lambda$ (reminiscent of the grid size distribution in the original MMAR), but are scattered randomly according to Poisson processes with frequencies of arrival depending inversely on the scale s, i.e. assuming $r_i = s_i^{-1}$ (instead of $r_i = 2^{i-k}$ at scales $s_i = 2^{k-i}$ over an interval $[0, 2^k]$ in the basic grid-bound approach for multifractal measures). Associating independent weights to the different scales one obtains a multifractal measure for this construction by taking a product of these weights over a conical[11] domain in (t, s) space. The theory of such cylindrical pulses (i.e., the pertinent multipliers $M^i_{t_n}$ that rule one hierarchical level between adjacent change points t_n and t_{n+1}) only needs the requirement of existence of $\mathbb{E}[M^i_{t_n}]$. Barral and Mandelbrot (2002) work out the "multifractal apparatus" for such more general families of hierarchical cascades pointing out that many examples of pertinent processes would be characterized by non-existing higher moments. Muzy and Bacry (2002) and Bacry and Muzy (2003) go one step further and construct a "fully continuous" class of multifractal measures in which the discreteness of the scales i is replaced by a continuum of scales.

[11] The conical widening of the influence of scales being the continuous limit of the dependencies across levels in the discrete case that proceeds with, e.g., a factor 2 in the case of binary cascades.

Multiplication over the random weights is then replaced by integration over a similar conical domain in (t,s) space whose extension is given by the maximum correlation scale T (see below). Muzy and Bacry (2002) show that for this set-up, nontrivial multifractal behavior is obtained if the conical subset $C_s(t)$ of the (t,s)-half plane (note that $t \geq 0$) obeys:

$$C_s(t) = \{(t', s'),\ s' \geq s,\ -f(s')/2 \leq t' - t \leq f(s')/2\} \tag{13}$$

with

$$f(s) = \begin{cases} s & \text{for } s \leq T \\ T & \text{for } s > T, \end{cases} \tag{14}$$

i.e. a symmetrical cone around current time t with linear expansion of the included scales s up to some maximum T. The multifractal measure obtained along these lines involves a stochastic integral over the domain $C(t)$:

$$d\theta(t) = e^{\int_{(t',s) \in C(t)} d\omega(t',s)}. \tag{15}$$

If $d\omega(t', s)$ is a Gaussian variable, one can use this approach as an alternative way to generate a Lognormal multifractal time transformation. As demonstrated by Bacry and Muzy (2003) subordinating a Brownian motion to this process leads to a compound process that has a distribution identical to the limiting distribution of the grid-bound MMAR with Lognormal multipliers for $k \to \infty$. Discretization of the continuous-time multifractal random walk will be considered below.

3.3 Multifractal Models in Discrete Time[12]

3.3.1 Markov-Switching Multifractal Model

Together with the continuous-time Poisson multifractal, Calvet and Fisher (2001) have also introduced a discretized version of this model, that has become the most frequently applied version of the multifractal family in the empirical financial literature. In this discretized version, the volatility dynamics can be interpreted as a discrete-time Markov-switching process with a large number of states. In their approach, returns are modeled like in *eq.* (1) with innovations ε_t drawn from a standard Normal distribution $N(0, 1)$ and instantaneous volatility being determined

[12] We note in passing that for standard discrete volatility models, the determination of the continuous-time limit is not always straightforward. For instance, for GARCH(1,1) model Nelson (1990) found a limiting "GARCH diffusion" under some assumptions while Corradi (2000) found a limiting deterministic process under a different set of assumptions. Also, while there exists a well-known class of continuous-time stochastic volatility models, these do not necessarily constitute the limit processes of their also well-known discrete counterparts.

by the product of k volatility components or multipliers $M_t^{(1)}, M_t^{(2)}, \ldots, M_t^{(k)}$ and a constant scale factor σ:

$$r_t = \sigma_t \varepsilon_t \tag{16}$$

with

$$\sigma_t^2 = \sigma^2 \prod_{i=1}^{k} M_t^i. \tag{17}$$

The volatility components M_t^i are persistent, non-negative and satisfy $E[M_t^i] = 1$. Furthermore, it is assumed that the volatility components $M_t^{(1)}, M_t^{(2)}, \ldots, M_t^{(k)}$ at a given time t are statistically independent. Each volatility component is renewed at time t with probability γ_i depending on its rank within the hierarchy of multipliers and remains unchanged with probability $1 - \gamma_i$. They show that with the following specification of transition probabilities between integer time steps, a discretized Poisson multifractal converges to the continuous-time limit as defined above for $\Delta t \to 0$:

$$\gamma_i = 1 - (1 - \gamma_1)^{(b^{i-1})}, \tag{18}$$

with γ_1 the component at the lowest frequency that subsumes the Poisson intensity parameter λ, $\gamma_1 \in [0, 1]$, and $b \in (1, \infty)$. Calvet and Fisher (2004) assume a Binomial distribution for M_t^i with parameters m_0 and $2 - m_0$ (thus, guaranteeing an expectation of unity for all M_t^i). If convergence to the limit of the Poisson multifractal is not a concern, one could also use a less parameterized form such as

$$\gamma_i = b^{-i}. \tag{19}$$

Here, volatility components in a lower frequency state will be renewed b times as often as those of its predecessor. An iterative discrete multifractal with such a progression of transition probabilities and otherwise identical to the model of Calvet and Fisher (2001, 2004) has already been proposed by Breymann *et al.* (2000).

For the distribution of the multipliers M_t^i, extant literature has also used the Lognormal distribution (cf. Liu, di Matteo and Lux, (2008); Lux, (2008)) with parameters λ and s, i.e.

$$M_t^{(i)} \sim LN(-\lambda, s^2). \tag{20}$$

Setting $s^2 = 2\lambda$ guarantees $E[M_t^i] = 1$. Comparison of the performance and statistical properties of MF models with Binomial and Lognormal multipliers shows typically almost identical results (cf. Liu, di Matteo and Lux (2007)). It, thus, appears that the Binomial choice (with 2^k different volatility regimes) has sufficient flexibility and cannot easily be outperformed via a continuous distribution of the multipliers.

In Fig. 5 the first three panels show the development of the switching behavior of Lognormal MSM process at different levels. The average duration of the second highest component is equal to 2048. As a result one expects this component to switch on average two times during the 4096 time-steps of the simulation. Similarly, for the sixth highest component displayed in the second panel renewal occurs about once within $2^5 = 32$ periods. The last panel shows the product of multipliers (displayed in the second from bottom) that plays the role of local stochastic volatility as described by *eq.* (17). The resulting artificial time series displays volatility clustering and outliers which stem from intermittent bursts of extreme volatility.

Due to its restriction to a finite number of cascade steps, the MSM is not characterized by asymptotic (multi-) scaling. However, its pre-asymptotic scaling regime can be arbitrarily extended by increasing the number of hierarchical components k. It is, thus, a process whose multifractal properties are spurious. However, at the same time it can be arbitrarily close to "true" multi-scaling over any finite length scale. This feature is shared by a second discretization, the multifractal random walk, whose power-law scaling over a finite correlation horizon is already manifest in its generating process.

Figure 5 about here

3.3.2 Multifractal Random Walk

In the (econo-)physics literature, a different type of causal, iterative process has been developed more or less simultaneously, denoted the Multifractal Random Walk (MRW). Essentially, the MRW is a Gaussian process with built-in multifractal scaling via an appropriately defined correlation function. While one could use various distributions for the multipliers as the guideline for construction of different versions of MRW replicating their particular autocorrelation structures, the literature has exclusively focused on the Lognormal distribution.

Bacry *et al.* (2001) define the MRW as a Gaussian process with a stochastic variance as follows:

$$r_{\Delta t}(\tau) = e^{\omega_{\Delta t}(\tau)} \varepsilon_{\Delta t}(\tau), \tag{21}$$

with Δt a small discretization step, $\varepsilon_{\Delta t}(\cdot)$ a Gaussian variable with mean zero and variance $\sigma^2 \Delta t$ and $\omega_{\Delta t}(\cdot)$ the logarithm of the stochastic variance and τ a multiple of Δt along the time axis. Assuming that $\omega_{\Delta t}(\cdot)$ also follows a Gaussian distribution, one obtains Lognormal volatility draws. For longer discretization steps (e.g. daily unit time intervals), one obtains their returns as:

$$r_{\Delta t}(t) = \sum_{i=1}^{t/\Delta t} \varepsilon_{\Delta t}(i) * e^{\omega_{\Delta t}(i)}. \tag{22}$$

To mimic the dependency structure of a Lognormal cascade, these are assumed to have covariances:

$$\mathbb{C}ov(\omega_{\Delta t}(t)\omega_{\Delta t}(t+h)) = \lambda^2 \ln \rho_{\Delta t}(h), \tag{23}$$

with

$$\rho_{\Delta t}(h) = \begin{cases} \frac{T}{(|h|+1)\Delta t}, & \text{for } |h| \leq \frac{T}{\Delta t} - 1 \\ 0, & \text{otherwise} \end{cases} \tag{24}$$

Hence, T is the assumed finite correlation length (a parameter to be estimated) and λ^2 is called the intermittency coefficient characterizing the strength of the correlation.

In order for the variance of $r_{\Delta t}(t)$ to converge, $\omega_{\Delta t}(\cdot)$ is assumed to obey:

$$\mathbb{E}(\omega_{\Delta t}(i)) = -\lambda^2 \ln(T/\Delta t) = -\mathbb{V}ar(\omega_{\Delta t}(i)). \tag{25}$$

Assuming a finite decorrelation scale (rather than a monotonic hyperbolic decay of the autocorrelation) serves to guarantee stationary of the multifractal random walk. Similar as the MSM introduced by Calvet and Fisher (2001), the MRW model does, therefore, not obey an exact scaling function like *eq.* (7) in the limit $t \to \infty$ or divergence of its spectral density at zero, but is characterized by only "apparent" long-term dependence over a bounded interval. The advantage of both models is that they possess "nice" asymptotic properties that facilitate application of many standard tools of statistical inference.

As shown by Muzy and Bacry (2002) and Bacry *et al.* (2008) the continuous-time limit of MRW (mentioned above in 3.2.1.3) can also be interpreted as a time transformation of a Brownian motion subordinate to a log-normal multifractal random measure. For this purpose, the MRW can be reformulated in a similar way like the MMAR model.

$$r(t) = B\left[\theta(t)\right], \quad \text{for all} \quad t \geq 0, \tag{26}$$

where $\theta(t)$ is a random measure for the transformation of chronological to "business time" and $B(t)$ is a Brownian motion independent of θ_t. "Business time" θ_t

is obtained along the lines of the above exposition of the MRW model as

$$\theta(t) = \lim_{\Delta \to 0} \int_0^t e^{2\omega_\Delta(u)} du. \tag{27}$$

Here $\omega_\Delta(u)$ is the stochastic integral of Gaussian white noise $dW(s,t)$ over a continuum of scales s truncated at the smallest and largest scales Δ and T which leads to a cone-like structure defining $\omega_\Delta(u)$ as the area delimited in time (over the correlation length) and a continuum of scales s in the (t,s) plane:

$$\omega_\Delta(u) = \int_\Delta^T \int_{u-s}^{u+s} dW(v,s) \tag{28}$$

To replicate the weight structure of the multipliers in discrete multifractal models, a particular correlation structure of the Gaussian elements $dW(v,s)$ needs to be imposed. Namely, the multifractal properties are obtained for the following choices of the expectation and covariances of $dW(v,s)$:

$$\mathbb{C}ov\left(dW(v,s), dW(v',s')\right) = \lambda^2 \delta(v-v')\delta(s-s')\frac{dvds}{s^2} \tag{29}$$

and

$$\mathbb{E}\left(dW(v,s)\right) = -\lambda^2 \frac{dvds}{s^2}. \tag{30}$$

Muzy and Bacry (2002) and Bacry and Muzy (2003) show that the limiting continuous-time process exists and possesses multifractal properties. Interestingly, Muzy *et al.* (2006) and Bacry *et al.* (2013) also provide results for the unconditional distribution of returns obtained from this process. They demonstrate that it is characterized by fat tails and that it becomes less heavy tailed under time aggregation. They also show that standard estimators of tail indices are ill-behaved for data from a MRW data-generating process due to the high dependency of adjacent observations. While the implied theoretical tail indices with typical estimated parameters of the MRW would be located at unrealistically large values (> 10), taking the dependency in finite samples into account one obtains biased (pseudo-)empirical estimates indicating much smaller values of the tail index that are within the order of magnitude of empirical ones. A similar mismatch between implied and empirical tail indices applies to other multifractal models as well (as far as we can see, this is not explicitly reported in extant literature, but has been mentioned repeatedly by researchers) and can be likely explained in the same way.

3.3.3 Asymmetric Univariate MF Models

All previous models are designed in a completely symmetric way for positive and negative returns. However, it is well known that price fluctuations in asset markets exhibit a certain degree of asymmetry due to leverage effects. The discrete-time skewed multifractal random walk (DSMRW) model proposed by Pochart and Bouchaud (2002) is an extended version of the MRW, that takes account of such asymmetries. The model is defined similarly as the MRW of *eq.* (21) but incorporates a direct influence of past realizations on contemporaneous volatility

$$\tilde{\omega}_{\Delta t}(i) \equiv \omega_{\Delta t}(i) - \sum_{k<i} K(k,i)\varepsilon_{\Delta t}(k), \tag{31}$$

where Pochart and Bouchaud propose to use $K(k,i) = \frac{K_0}{(i-k)^{\alpha}\Delta t^{\beta}}$ is a positive definite kernel for the influence of returns on subsequent volatility. Bacry *et al.* (2012) proposed a continuous-time skewed multifractal model that also incorporates the leverage effect.

Eisler and Kertész (2004) expand the MSM model in a similar way. They consider a refined version of the model in which asymmetry comes in via the renewal probabilities and, in addition, use a term inspired by *eq.* (31) to account for leverage autocorrelations.

An asymmetric MSM model has also been introduced by Calvet *et al.* (2013). They embed a multifractal cascade into a stochastic volatility model where the product of multipliers enters as a time-varying long-run anchor for the volatility dynamics while at the same time governing a jump component in returns that relates positive volatility shocks to negative return shocks.

3.3.4 Bivariate Multifractal Models

A bivariate MF model has first been introduced by Calvet *et al.* (2006). Consider a portfolio of two assets α and β. Let now denote r_t the vector of log-returns of the portfolio, and r_t^{α} and r_t^{β} the individual log-returns of the two assets, respectively. Following Calvet *et al.* the return of the portfolio is modeled as:

$$r_t = [g(M_t)]^{1/2} * \varepsilon_t, \tag{32}$$

where $g(M_t)$ denotes a 2×1 vector $M_{1,t} * M_{2,t} * \cdots * M_{k,t}$, $*$ denotes element by element multiplication and the column vectors $\varepsilon_t \in R^2$ are *i.i.d.* Gaussian

$N(\mathbf{0}, \mathbf{\Sigma})$ with covariance matrix

$$\mathbf{\Sigma} = \begin{bmatrix} \sigma_\alpha^2 & \rho_\varepsilon \sigma_\alpha \sigma_\beta \\ \rho_\varepsilon \sigma_\alpha \sigma_\beta & \sigma_\beta^2 \end{bmatrix}. \tag{33}$$

ρ_ε represents the unconditional correlation between the residuals as the first source of correlation between both returns. The period t volatility state is characterized by a $2 \times k$ matrix $M_t = (M_{1,t}; M_{2,t}; \ldots; M_{k,t})$ and the vector of the components at the i^{th} frequency is $M_{i,t} = (M_{i,t}^\alpha \; M_{i,t}^\beta)$. The volatility vectors $M_{i,t}$ are non-negative and satisfy $E[M_{i,t}] = \mathbf{1}$, where $\mathbf{1} = (1,1)'$. Economic intuition behind the choice of the dynamics for each vector $M_{i,t}$ is that volatility arrivals are correlated but not necessarily simultaneous across markets. For this reason Calvet and Fisher allow arrivals across series to be linked by a correlation coefficient λ. Consider two random variables $I_{i,t}^\alpha$ and $I_{i,t}^\beta$ which are equal to 1 if each series $c \in \{\alpha, \beta\}$ is hit by an information arrival with probability γ_i, and equal to zero otherwise. Calvet and Fisher specified the arrival vector to be *i.i.d.* and assumed its unconditional distribution to satisfy three conditions. First, the arrival vector is symmetrically distributed: $(I_{i,t}^\alpha, I_{i,t}^\beta) \overset{d}{=} (I_{i,t}^\beta, I_{i,t}^\alpha)$. Second, the switching probabilities of both series are equal for each level i: $P(I_{i,t}^\alpha = 1) = P(I_{i,t}^\beta = 1) = \gamma_i$, with γ_i following *eq.* (18) as for univariate MSM. Third, there exists $\lambda \in [0,1]$ such that

$$P(I_{i,t}^\alpha = 1 | I_{i,t}^\beta = 1) = (1-\lambda)\gamma_i + \lambda.$$

These three conditions define a unique distribution of $(I_{i,t}^\alpha, I_{i,t}^\beta)$ whose joint switching probabilities can be easily determined. Note that the univariate dynamics of each series coincides with a univariate MSM model. Idier (2011) proposed an extension of the bivariate MSM model by considering a time dependent covariance for the vector of residuals $\rho_\varepsilon(t)$.

Liu (2008) considered a closely related bivariate multifractal model built upon the assumption that two time series have a certain number of joint cascade levels in common, while the remaining ones are chosen independently. The returns are, then, modeled as:

$$r_{q,t} = \left[\left(\prod_i^k M_{i,t} \right) \left(\prod_{l=k+1}^n M_{l,t} \right) \right]^{1/2} * \varepsilon_t, \tag{34}$$

where $q = 1, 2$ refers to the two time series, both having an overall number of n levels of their volatility cascades, and they share a number k of joint cascade levels which govern the strength of their volatility correlation. Obviously, the larger

k, the more correlation between the volatility dynamics of both series. After k joint multiplicators, each series has separate additional multifractal components. ε_t is defined as in *eq.* (32) to follow a bivariate standard Normal distribution with correlation parameter ρ_ε. This model can be seen as a special case of a slightly generalized version of Calvet *et al.* (2006) allowing for heterogeneity of the correlation of volatility innovations, λ_i, across hierarchical levels and choosing an extreme specification in that part of the $\lambda_i (1 \leq i \leq k)$ are equal to 1 and the remaining ones are equal to 0. Liu and Lux (2013) show that the distinction between different degrees of correlation between volatility innovations indeed improves the fit and performance of the bivariate MSM, but the extreme specification of Liu (2008) with alternation between full dependence and lack of correlation is dominated by a more flexible approach. Interestingly, whether high or low frequency components are more correlated differs between markets.

3.3.5 Higher dimensional multifractal models

The bivariate models presented above can be generalized for more than two assets in various ways. Liu (2008)'s approach can be generalized in a straightforward way to an N-dimensional asset returns process. If one assumes that the N time series share a number of j joint cascades that govern the strength of their volatility correlation, the correlation of volatility arrivals could be generalized to the case of an arbitrary number of assets without having to add new parameters in the volatility part of the model. Additional parameters would, then, only come in via the correlation of the Gaussian innovations. If such a specification appears insufficient to capture the heterogeneity in return fluctuations across assets, one could consider a generalized framework with asset-specific multifractal parameter, m_0 or λ in the Binomial or Lognormal setting, respectively.

A generalization of the MRW in a similar vein had already been proposed by Bacry *et al.* (2000). They suggest to extend the MRW model to a multivariate Multifractal Random Walk (MMRW) in order to model portfolio behavior. Let $\boldsymbol{X}_t$ be a MMRW, then following Bacry *et al.* $\boldsymbol{X}_t$ is defined as:

$$\boldsymbol{X}(t) = \lim_{t \to 0} \boldsymbol{X}_{\Delta t}(t) = \lim_{t \to 0} \sum_{k=1}^{t/\Delta t} \epsilon_{\Delta t}[k] * e^{\omega_{\Delta t}[k]}, \tag{35}$$

where $\epsilon_{\Delta t}$ is now a vector of Gaussians with zero mean and variance-covariance function at lag τ $Cov(\epsilon_{i,\Delta t}(t), \epsilon_{j,\Delta t}(t+\tau)) = \delta(\tau)\boldsymbol{\Sigma}_{ij}\Delta t$. The magnitude process $\omega_{\Delta t}(\cdot)$ is also Gaussian with covariance $Cov(\omega_{i,\Delta t}(t), \omega_{j,\Delta t}(t+\tau)) = \boldsymbol{\Lambda}_{ij} \ln(T_{ij}/(\Delta t + |\tau|))$ for $(\Delta t + |\tau| < T_{ij})$ and 0 elsewhere. The matrix $\boldsymbol{\Lambda}$, labeled "multifractal matrix ", controls the non-linearity of the multifractal spectrum, and T_{ij} are different correlation lengths for the autocovariances and cross-covariances characterizing the process.

4 Estimation and Forecasting

Availability of efficient estimation procedures is essential for the application of theoretical asset-pricing models for practical purposes. The non-standard format of multifractal models has initially cast doubts on the applicability of many well-known statistical tools to this new family of volatility models. Fortunately, the members of the second generation multifractal models (MSM and MRW) seemed to be much more well-behaved (and have partially be designed to be so) in terms of asymptotic statistical behavior. Most effort has been spent so far to find stable and efficient inference methods for the discrete time MSM model with discrete or continuous distributions for multipliers or volatility components. In the following we present the estimation methods most often applied for MF models. We dispense with the traditional $f(\alpha)$ and $\tau(q)$ approach to inference which has been covered in detail in sec. 2.4. As it soon turned out in the pertinent literature when starting to adapt multifractal models to finance, the scaling-approach provides potentially very biased and volatile estimates in applications to financial data, and due to their fat tails, would even indicate existence of multifractal structure after randomization of such time series. The quest for more appropriate statistical methods has been motivated to a large extent by these deficiencies. The development of the Markov-switching multifractal model and the multifractal random walk have brought forward stochastic processes with more "convenient" asymptotic properties than their predecessors. As a consequence, they allow application of many established tools of inference. Nevertheless, their proximity to genuine long-memory might still be a concern and motivates to exert caution in empirical applications (e.g., while theoretical convergence of estimates might be trivially guaranteed, the pre-asymptotic regime might be much more extended than with other models).

4.1 Maximum Likelihood Estimation

Exact ML estimation has been primarily developed for the discrete-time MSM model with a discrete distribution for the volatility components or multipliers. Calvet and Fisher (2004) introduced an ML estimation approach for the Binomial Markov-switching multifractal (BMSM) model. To show how to perform ML estimation in this context, note that the log-likelihood function for a series of observations $\{r_t\}_{t=1}^T$ in its most general form may be expressed as:

$$L(r_1, \ldots, r_T; \varphi) = \sum_{t=1}^{T} \ln g(r_t | r_1, \ldots, r_{t-1}; \varphi), \tag{36}$$

where $g(r_t|r_1, \ldots, r_{t-1}; \varphi)$ is the likelihood function of the Markov-switching multifractal model, and φ is the vector of parameters. For Markov-switching models, the likelihood function can be decomposed in the following way: $g(r_t|r_1, \ldots, r_{t-1}; \varphi) = \omega_t(r_t|M_t = m^i, \varphi)(\pi_{t-1}A)$. The three components are defined as follows: $\omega_t(r_t|M_t = m^i, \varphi)$ is a vector of dimension 2^k of conditional densities of any observation r_t for volatility regimes m^i and A is the transition matrix which has components $A_{ij} = P(M_{t+1} = m^j|M_t = m^i)$. The last component within the likelihood function above is π_t, which is the vector of conditional probabilities of the volatility states given observations $\pi_t^i = P(M_t = m^i|r_1, \ldots, r_t; \varphi)$. The conditional probabilities can be recursively obtained through Bayesian updating

$$\pi_t = \frac{\omega_t(r_t|M_t = m^i, \varphi) * (\pi_{t-1}A)}{\sum \omega_t(r_t|M_t = m^i, \varphi) * (\pi_{t-1}A)}. \tag{37}$$

Different distributional assumptions for innovations could be embedded in this framework. The parameter vector of the BMSM with Gaussian innovations would be given by $\varphi = (m_0, \sigma)'$, while the parameter vector of a BMSM with Student-t innovations would be $\varphi = (m_0, \sigma, \nu)'$ where $\nu \in (2, \infty)$ is the distributional parameter accounting for the degrees of freedom in the density function of the Student-t distribution. The Student$-t$ distribution for return innovations has been used by Lux and Morales-Arias (2010) in order to enhance out-of-sample forecasts of the MSM model because it may allow the MSM model to better distinguish between volatility dependence and fat-tailed innovations.

An advantage of the ML procedure is that, as a by-product, it allows one to obtain optimal forecasts via Bayesian updating of the conditional probabilities $\pi_t = P(M_t = m^i|r_1, \ldots, r_t; \varphi)$ for the unobserved volatility states m^i, $i = 1, \ldots, 2^k$. ML estimation provides good precision in finite samples (cf. Calvet and Fisher (2004)). The closed-form solutions obtained in Lux (2013) for the transient density of the continuous-time Poisson MMAR would also enable one to estimate its parameters via exact ML as long as the distribution of the multipliers is discrete.

Although the applicability of the ML algorithm greatly facilitates estimation of MSM models, it is restrictive in the sense that it is practically feasible only for discrete distributions of the multipliers and, therefore, is not applicable for e.g., the case of a Lognormal distribution. Due to the potentially large state space (we have to take into account transitions between 2^k distinct states), ML estimation also encounters practical bounds of its computational demands for specifications with more than about $k = 10$ volatility components in the Binomial case. For multivariate MF models, the applicability of the ML approach is even more constrained from the computational side: in the bivariate case the evaluation of its transition matrix with size $4^k \times 4^k$ becomes unfeasible for choices of about $k > 5$. There has also been a recent attempt to estimate the MRW model via a likelihood approach. Løvsletten and Rypdal (2012) develop an approximate maximum likelihood method for MRW using a Laplace approximation of the likelihood function.

4.2 Simulated Maximum Likelihood

This approach is more broadly applicable to both discrete and continuous distributions for multipliers. To overcome the computational and conceptional limitation of exact ML estimation, Calvet *et al.* (2006) developed a simulated ML approach. They propose a particle filter to numerically approximate the likelihood function. The particle filter is a recursive algorithm that generates independent draws $M_t^{(1)}, \ldots, M_t^{(N)}$ from the conditional distribution of π_t. At time $t = 0$, the algorithm is initiated by draws $M_0^{(1)}, \ldots, M_0^{(N)}$ from the ergodic distribution $\bar{\pi}$. For any $t > 0$, the particles $\{M_t^{(n)}\}_{n=1}^N$ are sampled from the new belief π_t. To this end, the formula (37) within the ML estimation algorithm is replaced by a Monte Carlo approximation in SML. This means that the analytical updating via the transition matrix, $\pi_{t-1}A$, is approximated via the simulated transitions of the particles. Disregarding the normalization of probabilities (i.e., the denominator), the formula (37) can be rewritten as

$$\pi_t^i \propto \omega_t(r_t|M_t = m^i; \varphi) \sum_{j=1}^{4^k} \mathrm{P}\left(M_t = m^i | M_{t-1} = m^j\right) \pi_{t-1}^j, \tag{38}$$

and due to the fact that $M_t^{(1)}, \ldots, M_t^{(N)}$ are independent draws from π_{t-1}, the Monte Carlo approximation has the following format:

$$\pi_t^i \propto \omega_t(r_t|M_t = m^i; \varphi) \frac{1}{N} \sum_{n=1}^{N} \mathrm{P}\left(M_t = m^i | M_{t-1} = M_{t-1}^{(n)}\right). \tag{39}$$

The approximation, thus, proceeds by simulating each $M_{t-1}^{(n)}$ one step forward to obtain $\hat{M}_t^{(n)}$ given $M_{t-1}^{(n)}$. This step only uses information available at date $t-1$, and must therefore be adjusted at time step t to account for the information contained in the new return. This is achieved by drawing N random numbers q from 1 to N with probability

$$P(q=n) \equiv \frac{\omega_t(r_t|M_t = \hat{M}_t^{(n)};\varphi)}{\sum_{n'=1}^{N}\omega_t(r_t|M_t = \hat{M}_t^{(n')};\varphi)} \tag{40}$$

The distribution of particles is, thus, shifted according to their importance at time t. With simulated draws $M_t^{(n)}$ the Monte Carlo (MC) estimate of the conditional density is

$$\hat{g}(r_t|r_1,\ldots,r_{t-1};\varphi) \equiv \frac{1}{N}\sum_{n=1}^{N}\omega_t(r_t|M_t = \hat{M}_t^{(n)};\varphi), \tag{41}$$

and the log-likelihood is approximated by $\sum_{t=1}^{T}\ln\hat{g}(r_t|r_1,\ldots,r_{t-1};\varphi)$. The simulated ML approach makes it feasible to estimate MSM models with continuous distribution of multipliers as well as univariate and multivariate Binomial models with too high a number of states for exact ML. Despite this gain in terms of different specifications of MSM models that can be estimated, the computational demands of SML are still considerable, particularly for high numbers of particles N.

4.3 GMM Estimation

Again, this is an approach that is, in principle, applicable for both discrete and continuous distributions for multipliers. To overcome the lack of practicability of ML estimation, Lux (2008) introduced a Generalized Method of Moments (GMM) estimator that is also universally applicable to all specifications of MSM processes (discrete or continuous distribution for multipliers, Gaussian, Student$-t$ or various other distributions for innovations). In particular, it can be used in all those cases where ML is not applicable or computationally unfeasible. Its computational demands are also lower than those of SML and independent of the specification of the model. In the GMM framework for MSM models, the vector of parameters φ is obtained by minimizing the distance of empirical moments from their theoretical counterparts, i.e.

$$\hat{\varphi}_T = \arg\min_{\varphi\in\Phi} f_T(\varphi)' A_T f_T(\varphi), \tag{42}$$

with Φ the parameter space, $f_T(\varphi)$ the vector of differences between sample moments and analytical moments, and A_T a positive definite and possibly random weighting matrix. Moreover, $\hat{\varphi}_T$ is consistent and asymptotically Normal if suitable "regularity conditions" are fulfilled (cf. Harris and Mátyás (1999)) which are satisfied routinely for Markov processes.

In order to account for the proximity to long memory characterizing MSM models, Lux (2008) proposed to use log differences of absolute returns together with the pertinent analytical moment conditions, i.e.

$$\xi_{t,T} = \ln|r_t| - \ln|r_{t-T}|. \tag{43}$$

The above variable only has nonzero auto-covariances over a limited number of lags. To exploit the temporal scaling properties of the MSM model, covariances of various moments over different time horizons are chosen as moment conditions, i.e.

$$\mathrm{Mom}(T, q) = E\left[\xi^q_{t+T,T} \cdot \xi^q_{t,T}\right], \tag{44}$$

for $q = 1, 2$ and different horizons T together with $E[r_t^2] = \sigma^2$ for identification of σ in the MSM model with Normal innovations. In the case of the MSM-t model, Lux and Morales-Arias (2010) consider additional moment conditions in addition to (44), namely, $E[|r_t|]$, $E[|r_t^2|]$, $E[|r_t^3|]$, in order to extract information on the Student$-t$'s shape parameter.

Bacry *et al.* (2008) and Bacry *et al.* (2013) also apply the GMM method for estimating the MRW parameters (λ, σ, and T) using similar moments as in Lux (2008). Sattarhoff (2010) refines the GMM estimator for the MRW using a more efficient algorithm for the covariance matrix estimation. Liu (2008) adopts the GMM approach to bivariate and trivariate specifications of the MSM model. Leövey (2013) develops a simulated method of moments (SMM) estimator for the continuous-time Poisson multifractal model of Calvet and Fisher (2001).

Related work in statistical physics has recently also considered simple moment estimators for extraction of the multifractal intermittency parameters from data of turbulent flows (Kiyono *et al.*, 2007). Leövey and Lux (2012) compare the performance of a GMM estimator for multifractal models of turbulence with various heuristic estimators proposed in the pertinent literature, and show that the GMM approach typically provides more accurate estimates due to its more systematic exploitation of information contained in various moments.

4.4 Forecasting

With ML and SML estimates, forecasting is straightforward: With ML estimation, conditional state probabilities can be iterated forward via the transition matrix to deliver forecasts over arbitrarily long time horizons. The conditional probabilities of future multipliers given the information set $\Im_t$, $\hat{\pi}_{t,n} = P(M_n|\Im_t)$, are given by

$$\hat{\pi}_{t,n} = \pi_t A^{n-t}, \quad \forall n \in \{t, \ldots, T\}. \tag{45}$$

In the case of SML, iteration of the particles provides an approximation to the predictive density. Since GMM does not provide information on conditional state probabilities, Bayesian updating is not possible and one has to supplement GMM estimation with a different forecasting algorithm. To this end, Lux (2008) proposes best linear forecasts (cf. Brockwell and Davis (1991), *ch.* 5) together with the generalized Levinson-Durbin algorithm developed by Brockwell and Dahlhaus (2004). Assuming that the data of interest (e.g., squared or absolute returns) follow a stationary process $\{Y_t\}$ with mean zero, the best linear h-step forecasts are obtained as

$$\hat{Y}_{n+h} = \sum_{i=1}^{n} \phi_{ni}^{(h)} Y_{n+1-i} = \boldsymbol{\phi}_n^{(h)} \boldsymbol{Y}_n, \tag{46}$$

where the vectors of weights $\boldsymbol{\phi}_n^h = (\phi_{n1}^h, \phi_{n2}^h, \ldots, \phi_{nn}^h)'$ can be obtained from the analytical auto-covariances of Y_t at lags h and beyond. More precisely, $\boldsymbol{\phi}_n^{(h)}$ are any solution of $\boldsymbol{\Psi}_n \boldsymbol{\phi}_n^{(h)} = \boldsymbol{\kappa}_n^{(h)}$ where $\boldsymbol{\kappa}_n^{(h)} = (\kappa_{n1}^{(h)}, \kappa_{n2}^{(h)}, \ldots, \kappa_{nn}^{(h)})'$ denote the auto-covariances of Y_t and $\boldsymbol{\Psi}_n = [\kappa(i-j)]_{i,j=1,\ldots,n}$ is the variance-covariance matrix. In empirical applications, *eq.* (46) has been applied for forecasting squared returns as a proxy for volatility using analytical covariances to obtain the weights $\boldsymbol{\phi}_n^h$. Linear forecasts have also been used by Bacry *et al.* (2008) and Bacry *et al.* (2013) in connection with their GMM estimates of the parameters of the MRW model. Duchon *et al.* (2012) develop an alternative forecasting scheme for the MRW model in the presence of parameter uncertainty as a perturbation of the limiting case of an infinite correlation length $T \to \infty$.

5 Empirical Applications

Calvet and Fisher (2004) compare the forecast performance of the MSM model to those of GARCH, MS-GARCH, and FIGARCH models across a range of in-sample and out-of-sample measures of fit. Using four long series of daily exchange rates they find that at short horizons MSM shows about the same and sometimes a better performance than its competitors. At long horizons MSM more clearly outperforms all alternative models. Lux (2008) combines the GMM approach with best linear forecasts and compares different MSM models (Binomial MSM and Lognormal MSM with various numbers of multipliers) to GARCH and FIGARCH. Although GMM is less efficient than ML, Lux (2008) confirms the tendency of superior performance of MSM models over GARCH and FIGARCH in forecasting volatility of foreign exchange rates. Similarly promising performance in forecasting volatility and value-at-risk is reported for the MRW model by Bacry *et al.* (2008) and Bacry *et al.* (2013). Bacry *et al.* (2008) find that linear volatility forecasts provided by the MRW model outperform GARCH$(1,1)$ models. Furthermore, they show that MRW forecasts of the VaR at any time-scale and time-horizon are much more reliable than GARCH$(1,1)$ (Normal or Student$-t$) forecasts for both foreign exchange rates and stock indices.

Lux and Kaizoji (2007) investigate the predictability of both volatility and volume for a large sample of Japanese stocks. Using daily data of stock prices and trading volume available over 27 years (from 01/01/1975 to 12/31/2001), they examine the potential of time series models with long memory (FIGARCH, ARFIMA, multifractal) to improve upon the forecasts derived from short-memory models (GARCH for volatlity, ARMA for volume). For both volatility and volume, they find that the MSM model provides much safer forecasts than FIGARCH and ARFIMA and does not suffer from occasional dramatic failures as is the case with the FIGARCH model. This higher degree of robustness of MSM forecasts compared to alternative models is also confirmed by Lux and Morales-Arias (2013). They estimate the typical parameters of GARCH, FIGARCH, SV, LMSV and MSM models from a large sample of stock indices and compare the empirical performance of each model when applied to simulated data of any other model with typical empirical parameters. As it turns out, the MSM almost always comes in second best (behind the true model) when forecasting future volatility and even dominates combined forecasts from many models. It, thus, appears to be relatively safe for practitioners to use the MSM even if it were misspecified and another standard model were the "true" data-generating process.

Lux and Morales-Arias (2010) introduce the MSM model with Student-t innovations and compare its forecast performance to those of MSM models with Gaussian innovations, and (FI)GARCH. Using country data on all-share equity indices, government bonds and real estate security indices, they find that the MSM model with Normal innovations produces forecasts that improve upon historical volatility, but are in some cases inferior to FIGARCH with Normal innovations. By adding fat tails to both MSM and FIGARCH, they obtain improvements by MSM models for forecasting volatility while the forecast performance by FIGARCH deteriorates. They find also that one can obtain more accurate volatility forecasts by combining FIGARCH and MSM.

Lux *et al.* (2011) apply an adapted version of the MSM model to measurements of realized volatility. Using five different stock market indices (CAC 40, DAX, FTSE 100, NYSE Composite and S&P 500), they find that the realized volatility-Lognormal MSM model (RV-LMSM) model performs better than non-RV models

(FIGARCH, TGARCH, SV and MSM) in terms of mean-squared errors for most stock indices and at most forecasting horizons. They also point out that similar results are obtained in a certain number of instances when the RV-LMSM model is compared to the popular RV-ARFIMA model and forecast combinations of alternative models (non-RV and RV) could hardly improve upon forecasts of various single models.

Calvet *et al.* (2006) apply the bivariate model to the comovements of volatility of pairs of exchange rates. They find again that their model provides better volatility and value-at-risk (VaR) forecasts compared to the constant correlation GARCH (CC-GARCH) of Bollerslev (1990). Applying the refined bivariate MSM to stock index data, Idier (2011) confirms the results of Calvet *et al.* (2006). Additionally, he finds that his refined model shows significantly better performance than the baseline MSM and DCC models for horizons longer than ten days. Liu and Lux (2013) apply the bivariate model to daily data for a collection of bivariate portfolios of stock indices, foreign currencies and U.S. 1 Year and 2 Year Treasury Bonds. They find that the bivariate multifractal model generates better VaR forecasts than the CC-GARCH model, especially in the case of exchange rates, and that an extension allowing for heterogeneous dependency of volatility arrivals across levels improves upon the baseline specification both in in-sample and out-of-sample.

Chen *et al.* (2013) propose a Markov-switching multifractal duration (MSMD) model. In contrast to the traditional duration models inspired by GARCH-type dynamics, this new model uses the MSM process developed by Calvet and Fisher (2004),and thus can reproduce the long memory property of durations. By applying the MSMD model to duration data of twenty stocks randomly selected from the S&P 100 index and comparing it with the autoregressive conditional duration (ACD) model both in- and out-of-sample, they find that at short horizons both models yield about the same results while at long horizons the MSMD model dominates over the ACD model.

Baruník *et al.* (2012) independently developed a Markov-switching multifractal duration (MSMD) model whose specification is slightly different from that proposed by Chen *et al.* (2013). They also use the MSM process introduced by Calvet and Fisher (2004) as basic ingredient in the construction of the model. They apply the model to price durations of three major foreign exchange futures contracts and compare the predictive ability of the new model with those of the ACD model and long-memory stochastic duration (LMSD) model of Deo *et al.* (2006). They find that both LMSD and MSMD forecasts generally outperform the ACD forecasts in terms of the mean square error and mean absolute error. MSMD and LMSD models sometimes exhibit similar forecast performances, sometimes the MSMD model slightly dominates the LMSD model.

Segnon and Lux (2012) compared the forecast performance of Chen *et al.*'s (2013) MSMD model to those of the standard ACD and Log-ACD models with flexible distributional assumptions about innovations (Weibull, Burr, Lognormal and generalized gamma) using density forecast comparison suggested by Diebold *et al.* (1998) and the likelihood ratio test of Berkowitz (2001). Using data from eight stocks traded on the NYSE their empirical results speak in favor of superiority of the MSMD model. They also find that, in contrast to the ACD model, using flexible distributions for the innovations does not exert much of an influence on the forecast capability of the MSMD model.

Option price applications of multifractal models have started with Pochart and Bouchaud (2002) who show that their skewed MRW model could generate smiles in option prices. Leövey (2013) proposed a "risk-neutral" MSM process in order to extract the parameters of the MSM model from option prices. As it turns out, MSM models backed out from option data add significant information to those estimated from historical return data and enhance the forecast ability of future volatility.

Calvet *et al.* (2013*a*) proposed an extension of the continuous-time MSM process which in addition to the key properties of the basic MSM process also incorporates the leverage effect and dependence between volatility states and price jumps. Their model can be conceived as an extension of a standard stochastic volatility model in which long-run volatility is driven by shocks of heterogenous frequency that also trigger jumps in the return dynamics, and, so are responsible for negative correlation between return and volatility. They also develop a particle filter that permits the estimation of the model. By applying the model to option data they find that it can closely reproduce the volatility smiles and smirks. Furthermore, they also find that the model outperforms affine jump-diffusions and asymmetric GARCH-type models in- and out-of-sample by a sizeable margin.

Calvet *et al.* (2013*b*) developed a class of dynamic term structure models in which the number of parameters to be estimated is independent of the number of factors selected. This parsimonious design is obtained by a cascading sequence of factors of heterogenous durations that is modeled in the spirit of multifractal models. The sequence of mean reversion rates of these factors follows a geometric progression which is responsible for the hierarchical nature of the cascade in the model. In their empirical application to a bandwidth of LIBOR and swap rates, a cascade model with 15 factors provides a very close fit to the dynamics of the term structure and outperforms random walk and autoregressive specifications in interest rate forecasting.

Taken as a whole, the empirical studies summarized above provide mounting empirical evidence of the superiority of the MF over traditional GARCH models (MS-GARCH, FIGRACH) in terms of forecasting of long-term volatility and related tasks such a VaR assessment. In addition, the model appears quite robust, and has found successful applications in modeling of financial durations, the term structure of interest rates and option pricing.

6 Conclusion

The motivation for studying multifractal models for asset price dynamics derives from their built-in properties: since they generically lead to time series with fat tails, volatility clustering and different degrees of long-term dependence of power transformations of returns, they are able to capture all the universal "stylized facts" of financial markets. In the overview of extant applications above, MF-type models typically exhibit a tendency to perform somewhat better in volatility forecasting and VaR-assessment than the more traditional toolbox of GARCH-type models. Furthermore, multifractal processes appear to be relatively robust to misspecification, they seem applicable to a whole variety of variables of interest from financial markets (returns, volume, durations, interest rates) and are very directly motivated by the universal findings of fact tails, clustering of volatility and anomalous scaling. In fact, multifractal processes constitute the only known class of models in which anomalous scaling is generic while all traditional asset-pricing models have a limiting uni-scaling behavior. Capturing this stylized fact may, therefore, well make a difference - even if one can never be certain that multiscaling is not spuriously caused by an asymptotically unifractal model and although those multifractal models that have become the workhorse in empirical applications (MSM, MRW) are characterized themselves by only preasymptotic multiscaling.

Obviously, the introduction of multifractal models in finance did not unleash as much research activity as that of the GARCH or SV families of volatility models in the decades before. The overall number of contributions in this area is still relatively small and comes from a relatively small group of active researchers only. The reason for this abstinence might be that the first generation of multifractal models might have appeared clumsy and unfamiliar to financial economists. Their non-causal principles of construction along the dimension of different scales of a hierarchical structure of dependencies might have appeared too different from known iterative time series models hitherto applied. In addition, the underlying multifractal formalism (including scaling functions and distribution of Hölder exponents) had been unknown in economics and finance, and application of standard statistical methods of inference to multifractal processes appeared cumbersome or impossible. However, all these obstacles have been overcome with the advent of the second generation of multifractal models (MSM and MRW) that are statistically well-behaved and of an iterative, causal nature. Besides their promising performance in various empirical applications they even provide the additional advantage of having clearly defined continuous-time asymptotics so that applications in dicrete- and in continuous-time can be embedded in a consistent framework.

While the relatively short history of multifractal models in finance has already brought about a variety of specifications and different methodologies for statistical inference, some areas can be identified in which additional work should be particularly welcome and useful. These include: multivariate MF models, applications of the MF approach beyond the realm of volatility models such as the MF duration model, and its use in the area of derivative pricing.

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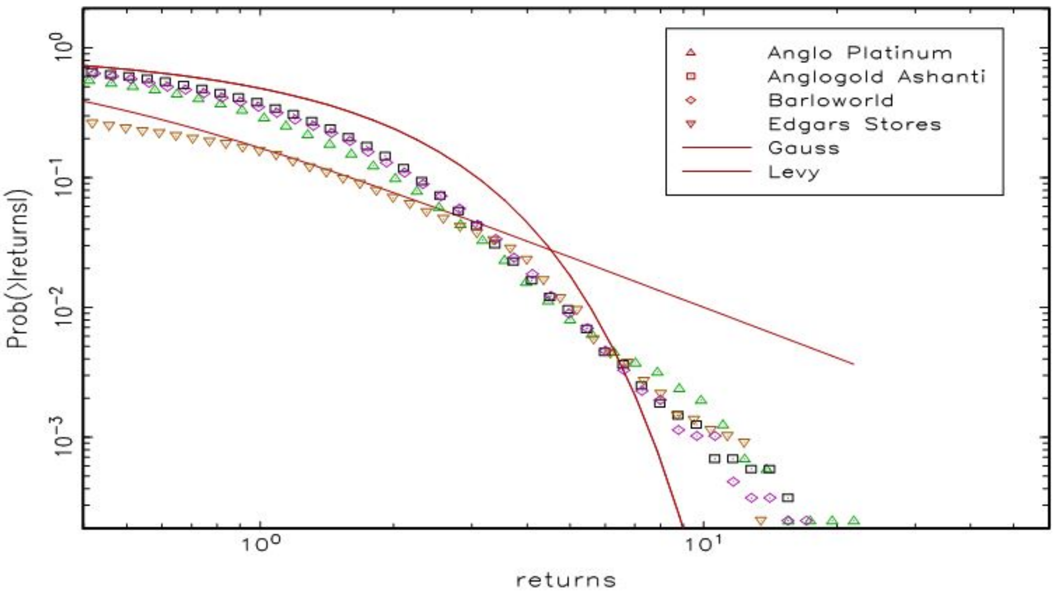

Figure 1: Cumulative distribution for daily returns of four South African stocks (from 1973 until 2006). The solid lines correspond to the Gaussian and Levy distributions. The tail behavior of all stocks is different from that of both the Gaussian and Levy distribution (for the latter, a characteristic exponent $\alpha = 1.7$ has been chosen that is a typical outcome of estimating the parameters of this family of distributions for financial data).

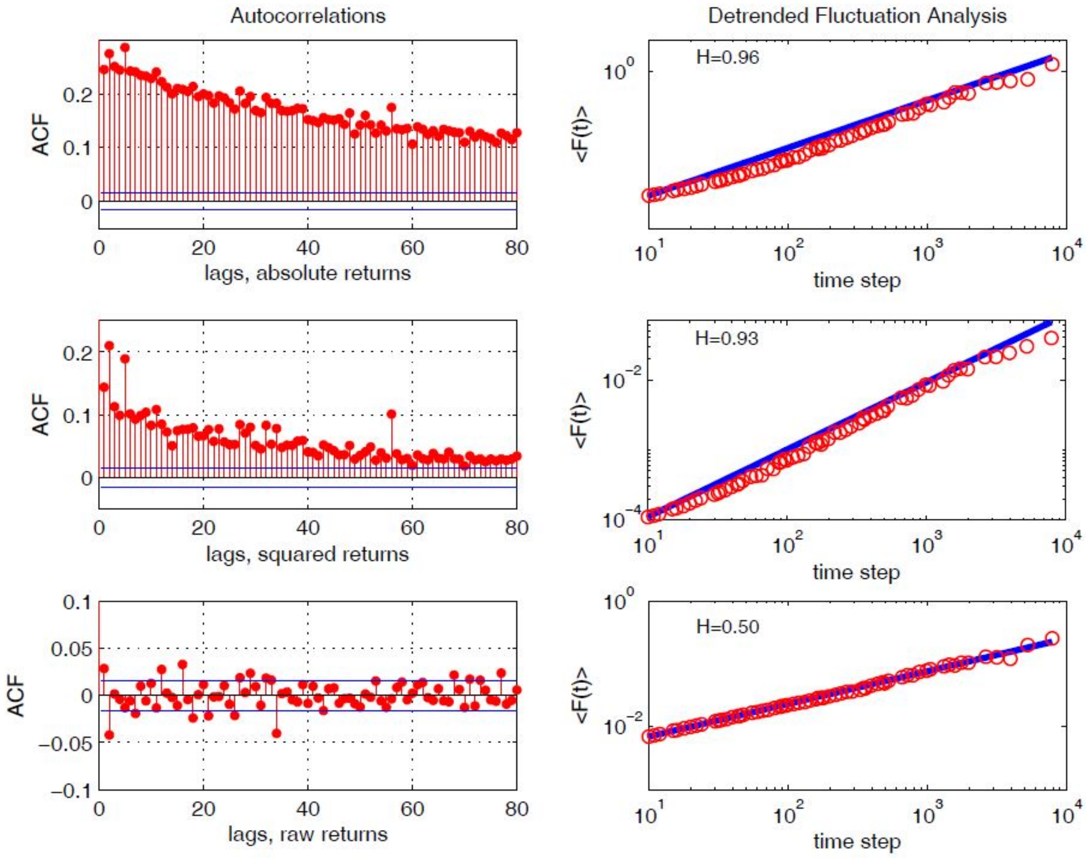

Figure 2: Illustration of the long-term dependence observed in the absolute and squared returns of the Standard & Poor's 500 index (S&P 500) (left upper and central panel). In contrast, raw returns (lower left panel) are almost uncorrelated. The determination of the corresponding Hurst exponent H via the so-called Detrended Fluctuation Analysis (DFA, cf. Chen *et al.* (2002)) is displayed in the right-hand panels. Note that we obtain the following scaling of the fluctuations (volatility): $< F(t) >\sim t^H$. $H = 0.5$ corresponds to absence of long-term dependency while $H > 0.5$ indicates a hyperbolical decay of the ACF, i.e. long-lasting autoregressive dependency.

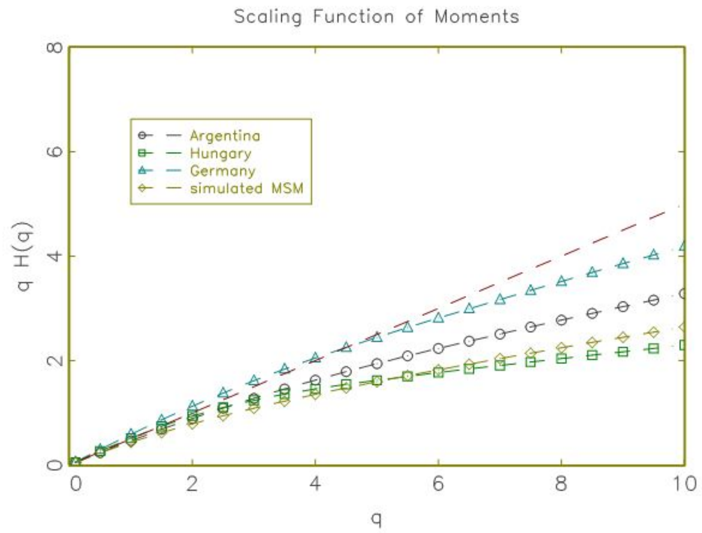

Figure 3: Scaling exponents of moments for three selected financial time series and an example of simulated returns from an MSM process. The empirical samples run from 1998 to 2007, and the simulated series is the one depicted in the lower panel of Fig. 5. The broken line gives the expected scaling $H(q) = q/2$ under Brownian motion. No fit has been attempted of the simulated to one of the empirical series.

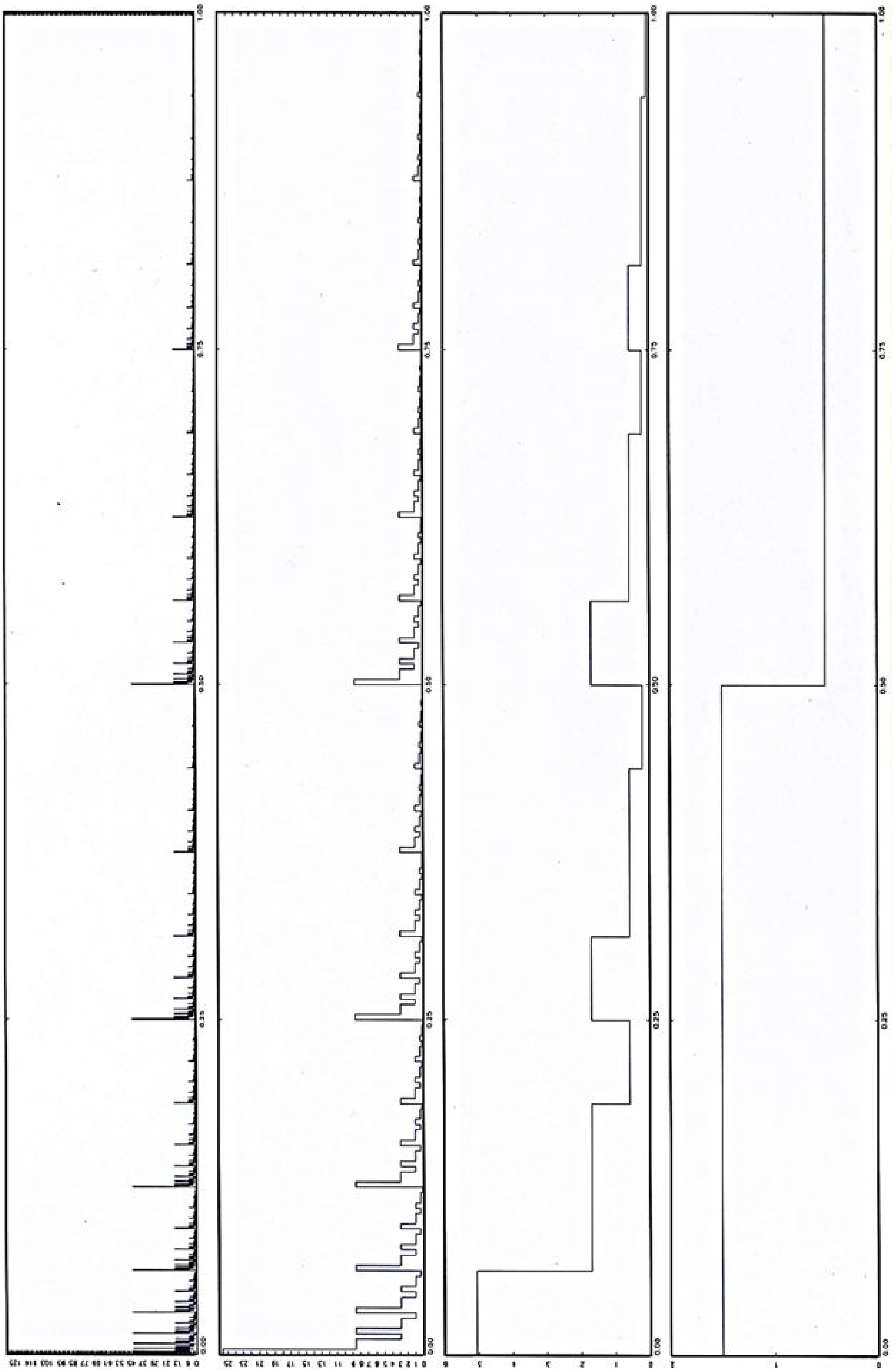

Figure 4: An illustration of the baseline Binomial multifractal cascade. Displayed are the resulting products of multipliers at steps 1, 4, 8 and 12. By moving to higher levels of cascade steps one observes a more and more heterogeneous distribution of the mass over the interval [0, 1].

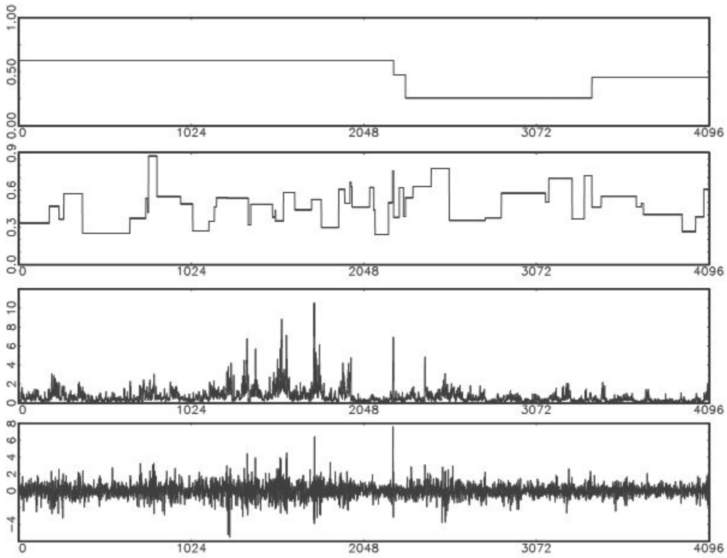

Figure 5: Simulation of a Markov-switching multifractal model (MSM) with Lognormal distribution of the multipliers and $k = 13$ hierarchical levels. The location parameter of the Lognormal distribution has been chosen as $\lambda = 1.05$. The first panel illustrates the development of the second multiplier (with average replacement probability of 2^{-11}), the second panel shows the sixth level, while the third panel shows the product of all 13 multipliers. Returns in the lowest panel are simply obtained by multiplying multifractal local volatility by Normally distributed increments.