On The Centrality of Policy Outcomes in Dynamic Majoritarian Bargaining Games

Tasos Kalandrakis
Department of Political Science, Yale University
PO Box 208301
New Haven, CT 06520-8301
E-mail: kalandrakis@yale.edu
URL: pantheon.yale.edu/~ak326/

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Abstract

We analyze an infinitely repeated divide-the-dollar bargaining game with an endogenous reversion point. In each session a new dollar is divided among three legislators according to the proposal of a randomly recognized member – if a majority prefer so – or according to previous period's allocation otherwise. We characterize a Markov Perfect Nash Equilibrium of this game and study the resultant equilibrium dynamics. Contrary to the results of Baron, 1996, for one-dimensional ideological spaces, or the intuition in Baron and Herron, 1999, for a finitely repeated analogue of the game over a two-dimensional space, outcomes in distributive policy spaces are considerably more extreme. Irrespective of the discount factor or the initial division of the dollar, equilibrium outcomes are absorbed in an irreducible set that has an empty intersection with the uncovered set (Miller, 1980, Epstein, 1998).

Keywords: Endogenous Reversion Point, Legislative Bargaining, Markov Perfect Nash Equilibrium, Stage Undominated Voting strategies, Uncovered Set.

– Comments Welcomed –

1. INTRODUCTION

We analyze the dynamics induced in an infinitely repeated legislative environment with a probabilistic recognition rule and an endogenous *reversion point* or *status quo*. In each period a legislator is randomly recognized to make a proposal for the allocation of a fixed renewable resource – a dollar – among members of the legislature. The proposed allocation is implemented if it receives a majority; otherwise, the resource is allocated as it was last period. We characterize a *Markov Perfect Nash Equilibrium* of this game, such that players condition their proposal and voting strategies only on previous period's decision.

Analysis of legislative games with recurring decisions and endogenous status quo (or reversion point) is limited in the literature. The difficulty arises from the fact that in such environments legislative decisions in the present have an impact on both the immediate as well as the future stream to benefits of players, rendering the strategic calculations involved – and hence characterization of equilibrium points – particularly challenging. Baron (1996) studies the same institutions as in the present analysis in a one-dimensional space of legislative outcomes where legislators have single-peaked 'stage' preferences. Applying the same equilibrium concept, he shows that legislative outcomes converge to the median from arbitrary initial policy decision.

Extension in higher-dimensional ideological spaces is considerably more difficult. Baron and Herron, 1999, analyze a finitely repeated version of this game that takes place over a two-dimensional space with three legislators and Euclidean stage preferences. This setting results in a badly behaved dynamic program for which the authors provide numerical solutions. They find that equilibrium legislative decisions tend to be more centrally located with a higher discount factor and a longer time horizon. These comparative statics lend credence to the conjecture that bargaining outcomes in the corresponding infinitely repeated game fall in a centrally located set, a result analogous to Baron's (1996) finding of (eventual) convergence to the median.

In a related exercise, Ferejohn, McKelvey, and Packel, (1984) consider the properties of the stochastic process induced over policy outcomes when a committee governed by majority rule with impatient or myopic members votes on alternatives that are randomly drawn from the winset[1] of the status quo – *i.e.* last period's decision as above. They show existence of a steady state distribution and provide numerical calculations where the bulk of the mass of the invariant distribution is concentrated in a centrally located subset of the policy space.

All these findings suggest that the set of outcomes reached in the long-run under an institutional setting of recurring legislative decisions that serve as reversion points for future deliberations is centrally located. Furthermore, they do not preclude the possibility that this set is related with solution concepts drawn from cooperative game theory (the *core*) or generalizations of the core when the latter is empty such as the *uncovered set* (Miller, 1980).

Our purpose in what follows is to show that this intuition does not carry through in a distributive, pork-barrel policy space. In particular, we characterize a Markov Perfect Equilibrium of the divide-the-dollar version of the game analyzed by Baron, 1996, for the case there are three legislators. Under this equilibrium, and irrespective of the initial allocation of the dollar or the discount factor, outcomes are absorbed with probability one in a set that consists of divisions that allocate the whole dollar to the proposer. Furthermore, this irreducible absorbing set has an *empty intersection with the uncovered set* (Epstein, 1998), thus providing a counterexample to the conjecture that the two sets are related via an inclusion property or overlap significantly.

The result can be motivated through the nature of winning coalitions that form along the equilibrium path. In the spirit of Riker (1962) coalitions are minimum winning (Riker, 1962) in that only a bare majority of members receive a positive fraction of the dollar when proposals are optimum. Less equitable allocations in the current period – such that excluded minorities receive zero share of the dollar – reduce the cost of building a coalition in subsequent periods since excluded committee members become less expensive. Thus, equilibrium dynamics constitute qualified analogues of the dynamics induced in the corresponding game for the case players are short-sighted or myopic. In fact, the steady-state, long-run distribution of policy outcomes under the characterized equilibrium is identical to

[1] The winset of alternative x is the set of points that are majority-preferred to x.

that induced when players place zero weight on future decisions.

Among positive findings, convergence to the equilibrium absorbing set of policy outcomes is fast, with a maximum expected time before absorption equal to two and a half (2.5) periods. Furthermore, only a finite number of allocations of the dollar (a set of measure zero in the space of possible allocations) are visited upon absorption, so that policy outcomes do not wander over the whole space of alternatives as interpretations of the various chaos theorems (McKelvey, 1976, 1979, Schofield, 1978, 1983) would posit. Nor do we observe perpetual instability since there is always positive probability that the same legislative decision prevails in consecutive periods once the steady state distribution of outcomes has been reached.

Kalandrakis (2001) generalizes these results to the case of more than three legislators and arbitrary (asymmetric) recognition probabilities and obtains additional insights with regard to optimal coalition building and equilibrium dynamics, as well as the role of more competitive agenda formation institutions. We now proceed to the presentation of the legislative setup. We present equilibrium analysis in section 3 and state the main result in section 4. We conclude in section 5.

2. LEGISLATIVE SETUP & EQUILIBRIUM NOTION

In the abstract, the problem involves a set $N = \{1, ..., n\}$ of $n > 2$ committee members that convene in periods $t \in \{1, 2, ...\}$ to choose a legislative outcome $\mathbf{x}^t \in X \subseteq \mathbf{R}^q$, X compact and $q \geq 1$, for each $t = 1, 2, .., +\infty$. At the beginning of each period legislator i is recognized with probability $p_i \geq 0$, $\forall i$, $\sum_{i=1}^{n} p_i = 1$ to make a proposal $\mathbf{z}$. Having observed the proposal legislators vote *yes* or *no*. If $m < n$, $m > 1$ or more vote *yes* then $\mathbf{x}^t = \mathbf{z}$; otherwise $\mathbf{x}^t = \mathbf{x}^{t-1}$. Thus, in the terminology of Romer and Rosenthal (1978) previous period's decision $\mathbf{x}^{t-1}$ serves as the *status quo* or *reversion point* in the current period t. Legislators derive vNM stage utility $u_i : X \longrightarrow \mathbf{R}$, $i \in N$, from the implemented proposal $\mathbf{x}^t$, with u_i continuous and bounded. The future is discounted by a factor $\delta_i \in [0, 1)$, so that the utility of legislator i from a sequence of legislative outcomes $\{\mathbf{x}^t\}_{t=1}^{+\infty}$ is given by:

$$U_i\left(\{\mathbf{x}^t\}_{t=1}^{+\infty}\right) = \sum_{t=1}^{+\infty} \delta_i^{t-1} u_i\left(\mathbf{x}^t\right) \tag{1}$$

Strategies in this game are functions that map *histories*[2] to the space of proposals X and voting decisions $\{yes, no\}$. In what follows, though, we restrict analysis to cases when players condition their behavior only on a summary of the history of the game that accounts for *payoff-relevant* effects of past behavior (see Fudenberg and Tirole, ch. 13). Specifically, let the *state* $\mathbf{s} \in S$ in period t be defined by previous period's allocation, *i.e.* $\mathbf{s} = \mathbf{x}^{t-1}$, $S = X$. Denote the space of Borel probability measures on X by $\wp(X)$. A (mixed) *proposal strategy* for legislator i – $\mu_i[\mathbf{z} \mid \mathbf{s}] \in \wp(X)$ – represents a probability distribution over legislative outcomes proposed by legislator i when recognized conditional on the state being $\mathbf{s}$; and a *voting strategy* is a measurable *acceptance set* $A_i(\mathbf{s}) \equiv \{\mathbf{z} \in X \mid i \text{ votes } yes \text{ if state is } \mathbf{s}\}$ for legislator i over proposals $\mathbf{z}$. Denote a (mixed) *Markov strategy* for legislator i by $\sigma_i(\mathbf{s}) = (\mu_i[\mathbf{z} \mid \mathbf{s}],\ A_i(\mathbf{s}))$. Restricting[3] analysis to such Markov strategies amounts to the requirement that players behave identically in different periods with the same state, even if that state arises from different histories.

To complete the statement of the equilibrium solution concept, define the *winset* of $\mathbf{x} \in X$ as:

$$W(\mathbf{x}) = \left\{\mathbf{y} \in X \mid \sum_{i=1}^{n} I_{A_i(\mathbf{x})}(\mathbf{y}) \geq m\right\} \tag{2}$$

with $I_B : X \longrightarrow \{0,1\}$ an indicator function $s.t. I_B(x) = \begin{cases} 1 & if \quad x \in B \\ 0 & if \quad x \in X \setminus B \end{cases}$. Given a n-tuple of Markov strategies $\sigma = \{\sigma_i\}_{i=1}^{n}$, we can define the *continuation value*, $v_i(\mathbf{s})$, of legislator i when the state is $\mathbf{s}$ as:

$$v_i(\mathbf{s}) = \int_X [u_i(\mathbf{x}) + \delta_i v_i(\mathbf{x})]\, Q[\mathbf{x} \mid \mathbf{s}]\, d\mathbf{x} \tag{3}$$

[2]A history is a vector that records all proposals as well as all voting decisions that precede an action (voting or proposing).

[3]If players play Markov strategies, then best responses are also Markov.

where $Q[\mathbf{x} \mid \mathbf{s}]$ denotes transition probabilities given by

$$Q[\mathbf{x} \mid \mathbf{s}] = \sum_{i=1}^{n} p_i I_{W(\mathbf{s})}(\mathbf{x}) \mu_i [\mathbf{x} \mid \mathbf{s}] + I_{\{s\}}(\mathbf{x}) \sum_{i=1}^{n} p_i \int_X I_{X \setminus W(\mathbf{s})}(\mathbf{y}) \mu_i [\mathbf{y} \mid \mathbf{s}] \, d\mathbf{y} \tag{4}$$

The first part of equation 4 reflects allocations that prevail when proposals obtain a majority, while the second part represents the probability that legislative policy decision is the same (i.e. the *reversion point* or *status quo* $\mathbf{s}$) as in the last period because the proposal did not receive a majority. On the basis of equation 3 re-write the (expected) utility of legislator i solely as a function of the current decision $\mathbf{x}^t$:

$$U_i(\mathbf{x}^t) = u_i(\mathbf{x}^t) + \delta_i v_i(\mathbf{x}^t) \tag{5}$$

where it is understood that $v_i(\mathbf{x}^t)$ – hence $U_i(\mathbf{x}^t)$ – are defined for given Markov strategies σ. Then:

Definition 1 *A Markov Perfect Nash Equilibrium in Stage-Undominated Voting strategies (MPNESUV) is a set of Markov strategies* $\sigma^* = \{\sigma_i^*\}_{i=1}^n = \{(\mu_i^*[\mathbf{z} \mid \mathbf{s}], \ A_i^*(\mathbf{s}))\}_{i=1}^n$, *such that* $\forall i \in N, \mathbf{s} \in S$*:*

$$\mathbf{y} \in A_i^*(\mathbf{s}) \Longleftrightarrow U_i(\mathbf{y}) \geq U_i(\mathbf{s}) \tag{Condition 1}$$

$$\mu_i^*[\mathbf{z} \mid \mathbf{s}] > 0 \Longrightarrow \mathbf{z} \in \arg\max \{U_i(\mathbf{x}) \mid \mathbf{x} \in W(\mathbf{s})\} \tag{Condition 2}$$

Baron, 1996, considers the case $X = \mathbf{R}_+$; u_i strictly concave and differentiable; n odd and $m = \frac{n+1}{2}$; and $\delta_i = \delta$, $p_i = p = \frac{1}{n}$, $\forall i$. We retain the last assumption focusing on the case $n = 3$, $m = 2$, and restrict legislative outcomes to the 2-dimensional unit simplex in $\mathbf{R}^3$, *i.e.* $\mathbf{x}^t = (x_1^t, x_2^t, x_3^t) \in X = \Delta$, with $x_i^t \geq 0, \sum_{i=1}^{3} x_i^t = 1$. We further assume risk neutral legislators that only care about their share of the dollar so that $u_i(\mathbf{x}) = x_i$, $i \in N$. Finally, given that legislators are otherwise identical, we focus on cases of symmetric *MPNESUV*, where players behave identically in identical situations up to arbitrary re-labeling of the policy outcome vector. To be precise, let $\pi(\mathbf{x})$ denote any permutation of the vector $\mathbf{x} \in \Delta$ and let $l(i; \pi(\mathbf{x})) : N \longrightarrow N$ be a 're-labeling' function that returns the position of coordinate x_i in $\pi(\mathbf{x})$. Then:

Definition 2 *In a symmetric MPNESUV* $\forall i \in N,\ \mathbf{s} \in S,$ *and* $\pi(\bullet)$*:*

$$\mathbf{x} \in A_i^*(\mathbf{s}) \Leftrightarrow \pi(\mathbf{x}) \in A_{l(i;\pi(\mathbf{s}))}^*(\pi(\mathbf{s})) \qquad \text{(Symmetry 1)}$$

$$\mu_i^*[\mathbf{z} \mid \mathbf{s}] = \mu_{l(i;\pi(\mathbf{s}))}^*[\pi(\mathbf{z}) \mid \pi(\mathbf{s})] \qquad \text{(Symmetry 2)}$$

3. EQUILIBRIUM ANALYSIS

Even in this considerably simplified setup, characterization of a MPNESUV constitutes a challenging problem due to the cardinality of the state space that makes it virtually impossible to ascertain the validity of equilibrium Condition 1 and Condition 2. The solution we present arises from an informative guess about the nature of the equilibrium-induced Markov process on policy outcomes defined in equation 4.

In particular, consider the case where equilibrium proposals involve 'minimum winning coalitions' (Riker, 1962) so that at most 2 legislators receive a positive fraction of the dollar in each period. Then certainly $s_i = 0$ for some i for all periods but the first. Further suppose that legislator i, with $s_i = 0$ does not object to new (optimal) divisions of the dollar $\mathbf{z}$ with $z_i = 0$, so that if $j \neq i$ is recognized in period $t+1$, a coalition of i and j vote *yes* on a proposal that allocates the whole dollar to j. But then both legislators i and $h \neq j$ receive zero, so that any of the three legislators can successfully form a coalition to extract the whole dollar in all subsequent periods.

If this conjectured path of play can be supported in equilibrium, then it is possible to solve this game backwards from the period when absorption to the set of outcomes that give zero to two legislators takes place, to arbitrary initial allocation of the dollar. It is by means of this strategy that we demonstrate the advertised result. Although the construction of this MPNESUV is not mathematically demanding the analysis becomes tedious due to the multiplicity of cases, as the reader can ascertain by the lengthy statement of the equilibrium – Proposition 1 – in the Appendix. Thus, rather than present a complete exposition of the analysis, in this section we offer a brief description of some basic steps that can prove enlightening both as to the nature of the solution and the process via which it is derived.

Additional notation will be necessary before we can proceed. First, partition the space of policy outcomes into subsets $\Delta_\theta \subset \Delta$, where $0 \leq \theta < 3$ indicates the number of legislators receiving zero share of the dollar:

$$\Delta_\theta = \left\{ \mathbf{x} \in \Delta \mid \sum_{i=1}^{n} I_{\{0\}}(x_i) = \theta \right\} \tag{6}$$

In the following three subsections we will describe equilibrium proposals for the cases θ is equal to 0, 1, and 2, respectively. We will illustrate how continuation values can be derived on the basis of these proposal – and voting – strategies. We note that throughout we assume – and only prove in the Appendix – that these proposals achieve majority passage and constitute optima for the proposers.

i. Recurrent Allocations: $\mathbf{s} \in \Delta_2$

According to the conjectured equilibrium, Δ_2 is an irreducible absorbing set. In particular, let generic elements of Δ_2 be denoted by $\mathbf{e}^i = \left(e_1^i, e_2^i, e_3^i\right)$, with $e_i^i = 1,\ e_j^i = 0,\ j \neq i$ and assume:

$$\mu_i^* \left[\mathbf{e}^i \mid \mathbf{s}\right] = 1, \forall i \in N, \mathbf{s} \in \Delta_2 \tag{7}$$

i.e. proposers always obtain the whole dollar when any one of the three legislators received the whole dollar in the previous period. The equilibrium-induced Markov process within this subset of the two dimensional simplex in $\mathbf{R}^3$ is depicted graphically in FIGURE 1a <<INSERT FIGURE 1 ABOUT HERE>>.

To obtain the continuation value of players for state, $\mathbf{s}$, within this subset, note that by substituting from equation 7 into 3 we obtain:

$$v_i(\mathbf{s}) = \sum_{j=1}^{3} \frac{1}{3} \left[e_i^j + \delta v_i\left(\mathbf{e}^j\right)\right], \forall i \in N, \mathbf{s} \in \Delta_2 \tag{8}$$

and observe that 8 applies for all possible $\mathbf{s} \in \Delta_2$, so that we can solve for $v_i(\mathbf{s})$ recursively to get:

$$v_i(\mathbf{s}) = \overline{v} = \frac{1}{3(1-\delta)}, \forall i \in N, \mathbf{s} \in \Delta_2 \tag{9}$$

ii. Transient Allocations: $\mathbf{s} \in \Delta_1$

Moving backwards, consider states $\mathbf{s}$ for, or prior to, the period of transition into Δ_2. Start with the case $\mathbf{s} \in \Delta_1$ first, *i.e.* cases when a single player received zero in the previous period. Notice that as a consequence of the focus on symmetric MPNESUV according to definition 2, it is sufficient to characterize equilibrium Markov strategies for $\mathbf{s} = (s_1, s_2, s_3)$ with $s_1 \geq s_2 \geq s_3$ – *i.e.* $s_3 = 0$ in this case. A natural candidate for players 1 and 2 proposal strategies is:

$$\mu_i^* \left[\mathbf{e}^i \mid (s_1, s_2, 0)\right] = 1, i = 1, 2. \tag{10}$$

In other words, since legislator 3 seems to be the 'least expensive' coalition partner, players 1 and 2 obtain the consent of player 3 in order to pass a proposal that allocates them the whole dollar.

With regard to player 3, consider the case she chooses player 2 as her coalition partner. By further invoking symmetry and Equilibrium Condition 1 we can deduce that player 2 is indifferent between $\mathbf{s} = (s_1, s_2, 0)$ and a proposal $(0, s_2, s_1)$ so that we can conjecture:

$$\mu_3^* \left[(0, s_2, s_1) \mid (s_1, s_2, 0)\right] = 1. \tag{11}$$

On the basis of the above write the continuation values of players as follows:

$$v_1(\mathbf{s}) = \frac{1}{3}\left[1 + \delta\overline{v}\right] + \frac{1}{3}\left[0 + \delta\overline{v}\right] + \frac{1}{3}\left[0 + \delta v_1(0, s_2, s_1)\right] \tag{12a}$$

$$v_2(\mathbf{s}) = \frac{1}{3}\left[0 + \delta\overline{v}\right] + \frac{1}{3}\left[1 + \delta\overline{v}\right] + \frac{1}{3}\left[s_2 + \delta v_2(0, s_2, s_1)\right] \tag{12b}$$

$$v_3(\mathbf{s}) = \frac{1}{3}\left[0 + \delta\overline{v}\right] + \frac{1}{3}\left[0 + \delta\overline{v}\right] + \frac{1}{3}\left[s_1 + \delta v_3(0, s_2, s_1)\right] \tag{12c}$$

Symmetry implies that $v_1(0, s_2, s_1) = v_3(\mathbf{s})$, $v_2(0, s_2, s_1) = v_2(\mathbf{s})$, and $v_3(0, s_2, s_1) = v_1(\mathbf{s})$ so that, after substitution, equation 12b can be solved for $v_2(\mathbf{s})$ and equations 12a and 12c can be solved for $v_3(\mathbf{s})$ and $v_1(\mathbf{s})$ to obtain:

$$v_1(\mathbf{s}) = \frac{1}{3(1-\delta)} - \frac{\delta s_2}{(9-\delta^2)} \tag{13a}$$

$$v_2(\mathbf{s}) = \frac{1}{3(1-\delta)} + \frac{s_2}{(3-\delta)} \tag{13b}$$

$$v_3(\mathbf{s}) = \frac{1}{3(1-\delta)} - \frac{3s_2}{(9-\delta^2)} \tag{13c}$$

The above analysis hinges on the assumption (equation 11) that player 3 is better off choosing player 2 as her coalition partner rather than player 1. Even though this is intuitive, given that the stake player 1 has on the status-quo $\mathbf{s}$ is higher than that of player 2, it is not true for all values of s_1, s_2. To verify that this is the case calculate the amount legislator 1 *demands*[4] from player 3 in order to vote *yes* on a proposal that excludes legislator 2, assuming the game is subsequently played according to equation 11. Denote this amount by d_1; we have:

$$d_1 + \delta v_1 (d_1, 0, 1 - d_1) = s_1 + \delta v_1 (\mathbf{s}) \tag{14}$$

which, substituting for $s_1 = 1 - s_2$ as well as $v_1 (d_1, 0, 1 - d_1) = \frac{1}{3(1-\delta)} + \frac{d_1}{(3-\delta)}$ and $v_1 (\mathbf{s}) = \frac{1}{3(1-\delta)} - \frac{\delta s_2}{(9-\delta^2)}$ from 13b and 13a respectively, we can solve for $d_1$to get:

$$d_1 = \frac{\left(9 - \delta^2\right) - 9s_2}{3\,(3 + \delta)} \tag{15}$$

This is smaller than the amount demanded by legislator 2, s_2, when:

$$s_2 \geq \frac{\left(9 - \delta^2\right)}{3\,(6 + \delta)} \tag{16}$$

As a consequence, when the difference between s_1 and s_2 is small – *i.e.* equation 16 is satisfied – legislator 3 has an incentive to choose legislator 1 as her coalition partner. In particular, in equilibrium player 3 will propose $(s_2, 0, s_1)$ – instead of $(0, s_2, s_1)$ – a fraction of the time and $(0, s_2, s_1)$ in the remainder of cases. If we denote the mixing probabilities as $\mu_3^* = \mu_3^* [(s_2, 0, s_1) \mid \mathbf{s}] = 1 - \mu_3^* [(0, s_2, s_1) \mid \mathbf{s}]$, we can re-write equations 12a, 12b, and 12c as:

$$v_1 (\mathbf{s}) = \frac{1}{3} [1 + \delta\overline{v}] + \frac{1}{3}\delta\overline{v} + \frac{1}{3} [\mu_3^* [s_2 + \delta v_1 (s_2, 0, s_1)] + (1 - \mu_3^*)\, \delta v_1 (0, s_2, s_1)] \tag{17a}$$

$$v_2 (\mathbf{s}) = \frac{1}{3}\delta\overline{v} + \frac{1}{3} [1 + \delta\overline{v}] + \frac{1}{3} [\mu_3^* \delta v_2 (s_2, 0, s_1) + (1 - \mu_3^*) [s_2 + \delta v_2 (0, s_2, s_1)]] \tag{17b}$$

$$v_3 (\mathbf{s}) = \frac{1}{3}\delta\overline{v} + \frac{1}{3}\delta\overline{v} + \frac{1}{3} [s_1 + \delta v_3 (0, s_2, s_1)] \tag{17c}$$

[4] A precise definition of *demands* appears in the Appendix.

which, after substituting $v_1(0, s_2, s_1) = v_2(s_2, 0, s_1) = v_3(\mathbf{s})$, $v_2(0, s_2, s_1) = v_1(s_2, 0, s_1) = v_2(\mathbf{s})$, and $v_3(0, s_2, s_1) = v_1(\mathbf{s})$, and along with:

$$s_2 + \delta v_1(s_2, 0, s_1) = s_1 + \delta v_1(\mathbf{s}) \tag{18}$$

(since $d_1 = s_2$) form four linear equations in four unknowns that can be solved for:

$$\mu_3^*[(0, s_2, s_1) \mid \mathbf{s}] = \frac{3(6+\delta)s_2 - (9-\delta^2)}{\delta(3+2\delta)} \tag{19}$$

$$v_1(\mathbf{s}) = \frac{(15-\delta)}{6(1-\delta)(6+\delta)} - \frac{1-2s_2}{2\delta} \tag{20a}$$

$$v_2(\mathbf{s}) = \frac{(15-\delta)}{6(1-\delta)(6+\delta)} + \frac{1-2s_2}{2\delta} \tag{20b}$$

$$v_3(\mathbf{s}) = \frac{(3+4\delta)}{3(1-\delta)(6+\delta)} \tag{20c}$$

The equilibrium induced Markov Process described above is depicted graphically in Figure 1b. Note that absorption into Δ_2 occurs whenever legislators 1 and 2 are recognized, *i.e.* there is only $\frac{1}{3}$ probability – when legislator 3 is recognized – that the decision remains in Δ_1 each period $\mathbf{s} \in \Delta_1$.

iii. Transient Allocations: $t = 1, \mathbf{s} \in \Delta_0$

If proposers never allocate a positive fraction of the dollar to more than one other legislator, allocations with all three legislators having a positive amount never prevail except perhaps for the very first period. For the latter cases, when the game happens to start with a state $\mathbf{s} \in \Delta_0$, equilibrium proposals are no different in nature than those analyzed so far, except for the additional complexity introduced by the various combinations of mixed and pure proposal strategies for various subsets of Δ_0 (eight subcases in total). Thus we will limit ourselves to a discussion of the equilibrium in this case, with the detailed characterization appearing in the Appendix.

First, the pattern of mixed proposal strategies described in the case $\mathbf{s} \in \Delta_1$ is also a feature of the equilibrium whenever the difference in allocated amounts under the state

$\mathbf{s} \in \Delta_0$ between any pair of players is small. While players with larger amount under $\mathbf{s}$ are more expensive coalition partners *ceteris paribus*, they become more willing to vote *yes* on a proposal to overturn the *status quo* if they are certain – when pure proposal strategies are played – they are excluded from the winning coalition. Conversely, legislators certain of being included in the winning coalition become more expensive since voting *no* on the motion on the floor still implies a high probability of being included in the winning coalition subsequently. By mixing in these cases the proposer ensures that coalition partners with a less favorable allocation under $\mathbf{s}$, do not become too intransigent in their demands because they believe they are guaranteed a position in the winning coalition.

Figure 2a-d depicts the two-dimensional unit simplex in $\mathbf{R}^3$ where the highlighted areas show cases when such mixing between coalition partners takes place for alternative values of the discount factor <<INSERT FIGURE 2 ABOUT HERE>>. Note that this happens for pairs of players along lanes of equitable allocation between them, as well as in the center of the simplex for all pairs of legislators since all players receive nearly equal share of the dollar. From a comparison of the three graphs it is apparent that mixing takes place in a smaller fraction of the cases as δ decreases. This is a direct consequence of the fact that the weight players put in the future benefit/cost of a change in the status quo (probability of inclusion in the coalition) diminishes with δ.

Another feature of the equilibrium that is implicit in the above discussion and is illustrated in Figures 2a-d is the fact that players are willing to vote *yes* on proposals that allocate them a smaller share of the dollar than what they obtain under the state $\mathbf{s}$. In fact, there are areas – near the sides of the triangle – in which the player with the smallest share of the dollar accepts proposals that allocate her zero and the whole dollar to the proposer. These players take into account both the immediate loss in accepting a smaller amount than what they obtain under the status quo, $\mathbf{s}$, as well as the *externality* that such MWC proposals generate through a reduction in their coalition building costs in the future. Specifically, if recognized in the next period they are able to extract the whole dollar, while they would have to allocate a positive amount to one of the other players had they rejected the proposal and preserved the status quo. As is the case for mixed proposal strategies, the area of direct absorption to Δ_2 contracts with δ, since the value of future reduction in coalition building costs diminishes as well.

4. RESULTS

The Equilibrium induced Markov process analyzed above is depicted graphically in Figure 3 <<INSERT FIGURE 3 ABOUT HERE>>. While the Appendix contains the exact statement of the equilibrium, here is a summary of the main finding:

Summary 1 *For any $\delta \in [0, 1)$ there exists a symmetric MPNESUV that induces a Markov process over outcomes such that:*

- Δ_2 *is an irreducible absorbing set,*
- *There is probability $\frac{2}{3}$ of transition from an outcome in Δ_1 into Δ_2, and $\frac{1}{3}$ of remaining into Δ_1,*
- *For some $\mathbf{s} \in \Delta_0$ there is probability $\frac{2}{3}$ of transition into Δ_2, by a majority formed by the proposer and the player with minimum amount in $\mathbf{s}$, and probability $\frac{1}{3}$ of transition into Δ_1,*
- *In the remaining cases of states $\mathbf{s} \in \Delta_0$ there is probability 1 of transition into Δ_1, and*
- *For $\mathbf{s} \in \Delta_0, \Delta_1$ proposers mix between coalition partners when the latter have positive but nearly equal allocation under the state, $\mathbf{s}$.*

The work of Epstein, 1998, on distributive policy spaces allows a direct comparison of the set of outcomes that eventually prevail under the characterized MPNESUV, with the uncovered set (Miller, 1980)[5]. More precisely, if we denote the uncovered set of Δ by $UC(\Delta)$, then:

[5] Epstein uses the following definition of the covering relation: y covers x if $y \succ x$ and $z \succ y \Longrightarrow z \succ x$, where $\succ$ is the (strong) majority preference relation.

Corollary 1 *Irrespective of the initial allocation of the dollar or the discount factor, equilibrium decisions eventually fall outside the uncovered set of* Δ *with probability one, or* $\lim_{t \longrightarrow +\infty} P\left[\mathbf{x}^t \in UC(\Delta)\right] = 0$.

Proof. Elements in Δ_2 are the only covered alternatives in Δ (Epstein, 1998, theorem 2, p 88-89). ■

Although focus on the long-run distribution of policy outcomes is natural in such dynamic games, the significance of the above negative findings would be undermined if convergence to the steady state distribution was slow. If that were the case – and depending on the initial allocation of the dollar – legislative policy decisions might concentrate in an area of relatively equitable allocations for a significant period of time before eventual absorption. This is not the case, since – except perhaps for the very first period – there is probability $\frac{2}{3}$ of absorption into Δ_2, from any equilibrium allocation not in that set. As a result:

Corollary 2 *The maximum expected time before absorption to* Δ_2 *is 2.5 periods.*

Proof. Denote the number of periods before absorption by T. If the probability of absorption is ρ, then the expected time before absorption, $E[T] = \sum_{T=1}^{+\infty} \rho (1-\rho)^{T-1} T = \rho \sum_{T=1}^{+\infty} (1-\rho)^{T-1} T = \rho \frac{1}{\rho^2} = \frac{1}{\rho}$. Hence for $\rho = \frac{2}{3}$, $E[T] = \frac{3}{2}$ which is true for the case $\mathbf{s} \in \Delta_1$ and cases 3a-c of Proposition 1, when $\mathbf{s} \in \Delta_0$. In the remaining cases ($\mathbf{s} \in \Delta_0$, cases 3d-h) there is probability 1 of transition into Δ_1 so that $\max E[T] = 1 + \frac{3}{2} = 2.5$. ■

5. CONCLUSIONS

We analyzed a three-player majority rule bargaining game with a recurring decision over a distributive policy space and an endogenous reversion point or default alternative. We provided a complete characterization of a Markov Perfect Nash equilibrium for this dynamic game, where players condition their strategies only on previous period's decision; and we studied the properties of the Markov process on policy outcomes induced by this equilibrium. A number of theoretically significant insights into the workings of majoritarian bargaining emerge from this exercise.

On the one hand, existence of an equilibrium implies that committees or societies are able to reach decisions in dynamic multi-period contexts even over policy spaces for which majority rule induces a pathological social preference relation. Despite the all-encompassing nature of the social preference cycle in this environment (Mckelvey, 1976, 1979, Schofield, 1978, 1983), only a finite number of alternatives are reached with positive probability upon absorption to the steady-state distribution under the characterized equilibrium. Also, as shown in Corollary 2, convergence to this long-run distribution is fast. Finally, within this long-run equilibrium set, instability of decisions is of a stochastic nature only, since there is positive probability that the same decision prevails in any two consecutive periods.

Yet, unlike the finding of Baron, 1996, or the intuition arising from the simulations of Baron and Herron, 1999, and Ferejohn, McKelvey and Packel, 1984, policy outcomes are extreme in the sense that – with probability one – only the least equitable allocations eventually prevail. This is also true if we compare these – long-run – outcomes with the allocations that prevail in distributive policy spaces under the institutions considered by Baron and Ferejohn, 1989[6], since a majority of legislators receive a positive fraction of the dollar in their analysis. Furthermore, outcomes fall outside the uncovered set, thus providing a counterexample to the conjecture that the two sets are related via an inclusion property or display significant overlap.

These findings generate intuition for the common institutional choice – contrary to what is the case for legislation on ideological spaces – that monetary allocations (budgets or appropriations) in legislatures with endogenous agenda formation are deliberated under an exogenous (fixed) reversion point (often zero spending). Also, the non-centrality of policy outcomes compared to those that prevail in ideological spaces supports the remark of Epstein, 1998, who argues that institutional arrangements are more consequential in distributive spaces when the preferences of political agents are in direct conflict with each other, compared to ideological spaces.

[6]These institutions are analyzed in more general context by Banks and Duggan, 2000. Although the authors do not locate the set of equilibirum policy outcomes in general, they do show *core implementation* under quite general assumptions and upper-hemicontinuity of equilibrium outcomes.

While the rather narrow setup of the model (three legislators, distributive policy space, etc.), enabled us to provide a complete characterization of the equilibrium, a host of open questions await further exploration[7]. First, in light of the discussion in the previous paragraph, it is important to consider whether the non-centrality of policy decisions persists in legislative environments with more competitive agenda formation institutions, such as the open rule considered by Baron and Ferejohn, 1989. Also, while the equilibrium in this paper is not unique in the class of MPNESUV, all other equilibria we can characterize are payoff-equivalent[8], leaving open the question of general payoff-equivalence of the class of MPNESUV over distributive spaces. Finally, as discussed by Banks and Duggan, 2000, the existence and continuity properties of MPNESUV in more general policy spaces and decision rules constitutes a challenging unresolved problem.

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[7]Some of these questions I take up in Kalandrakis, 2001, where I generalize the analysis in this paper to legislatures with more than four members and asymmetric recognition probabilities.

[8]The multiplicity of equilibria arises from the payoff equivalence of allocations (hence multiplicity of payoff equivalent demands) under case 2b of Proposition 1.

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APPENDIX

The following definition is useful in the statement and proof of the equilibrium:

Definition 3 *Let the* demand *of legislator i for a state* $\mathbf{s}$*,* $d_i(\mathbf{s})$*, be equal to zero if*

$$U_i\left(\mathbf{d}^{ij}(\mathbf{s})\right) \geq U_i(\mathbf{s}) \tag{D1}$$

or an amount $d_i(\mathbf{s}) \in (0,1]$ *such that*

$$U_i\left(\mathbf{d}^{ij}(\mathbf{s})\right) = U_i(\mathbf{s}) \tag{D2}$$

otherwise, where $\mathbf{d}^{ij}(\mathbf{s}) \in \Delta$ *is a minimum-winning-consistent proposal with* $d_i^{ij}(\mathbf{s}) = d_i(\mathbf{s})$ *and* $d_j^{ij}(\mathbf{s}) = 1 - d_i(\mathbf{s}), j \neq i$*.*

Thus, when a legislator receives her *demand* and a third legislator receives zero, the legislator that receives her demand strictly or weakly prefers that proposal to $\mathbf{s}$. Note that demands $d_i(\mathbf{s})$ need not be unique if positive, but if they are not unique they determine minimum-winning-consistent proposals $\mathbf{d}^{ij}(\mathbf{s})$ that are payoff equivalent for legislator i. For notational simplicity in what follows we omit the dependence of demands and minimum-winning-consistent proposals on the state $\mathbf{s}$ and write d_i and $\mathbf{d}^{ij}$ instead, unless otherwise necessary.

Proposition 1 *There exists a symmetric MPNESUV with the following demands, proposal strategies, and continuation value functions for all* $\mathbf{s}$ *with* $s_1 \geq s_2 \geq s_3$*:*

- *Case 1:* $\mathbf{s} \in \Delta_2$

$$d_1 = 1, d_2 = d_3 = 0 \tag{21}$$

$$\mu_i^*\left[\mathbf{e}^i \mid \mathbf{s}\right] = 1, \forall i \tag{22}$$

$$v_i(\mathbf{s}) = \frac{1}{3(1-\delta)}, \forall i \tag{23}$$

- *Case 2a:* $\mathbf{s} \in \Delta_1$ *and* $s_2 \leq \frac{(9-\delta^2)}{3(6+\delta)}$

$$d_1 = \frac{\left(9-\delta^2\right) - 9s_2}{3\left(3+\delta\right)}, d_2 = s_2, d_3 = 0 \tag{24}$$

$$\mu_i^*\left[\mathbf{d}^{3i} \mid \mathbf{s}\right] = \mu_3^*\left[\mathbf{d}^{23} \mid \mathbf{s}\right] = 1,\ \ i = 1, 2 \tag{25}$$

$$v_1\left(\mathbf{s}\right) = \frac{1}{3\left(1-\delta\right)} - \frac{\delta s_2}{\left(9-\delta^2\right)} \tag{26a}$$

$$v_2\left(\mathbf{s}\right) = \frac{1}{3\left(1-\delta\right)} + \frac{s_2}{\left(3-\delta\right)} \tag{26b}$$

$$v_3\left(\mathbf{s}\right) = \frac{1}{3\left(1-\delta\right)} - \frac{3s_2}{\left(9-\delta^2\right)} \tag{26c}$$

- *Case 2b:* $\mathbf{s} \in \Delta_1$ *and* $s_2 > \frac{(9-\delta^2)}{3(6+\delta)}$

$$d_1 = d_2 = s_2, d_3 = 0 \tag{27}$$

$$\mu_i^*\left[\mathbf{d}^{3i} \mid \mathbf{s}\right] = 1, \quad i = 1, 2 \tag{28}$$

$$\mu_3^*\left[\mathbf{d}^{23} \mid \mathbf{s}\right] = 1 - \mu_3^*\left[\mathbf{d}^{13} \mid \mathbf{s}\right] = \frac{3\left(6+\delta\right)s_2 - \left(9-\delta^2\right)}{\delta\left(3+2\delta\right)} \tag{29}$$

$$v_1\left(\mathbf{s}\right) = \frac{\left(15-\delta\right)}{6\left(1-\delta\right)\left(6+\delta\right)} - \frac{1-2s_2}{2\delta} \tag{30a}$$

$$v_2\left(\mathbf{s}\right) = \frac{\left(15-\delta\right)}{6\left(1-\delta\right)\left(6+\delta\right)} + \frac{1-2s_2}{2\delta} \tag{30b}$$

$$v_3\left(\mathbf{s}\right) = \frac{\left(3+4\delta\right)}{3\left(1-\delta\right)\left(6+\delta\right)} \tag{30c}$$

- *Case 3a:* $\mathbf{s} \in \Delta_0, s_3 \leq \frac{3\delta}{(9-\delta^2)}s_2,\ s_3 \leq 1 - \frac{3(6+\delta)}{(9-\delta^2)}s_2$

$$d_1 = \frac{\left(9-\delta^2\right) - 9s_2}{3\left(3+\delta\right)}, d_2 = s_2, d_3 = 0 \tag{31}$$

$$\mu_i^*\left[\mathbf{d}^{3i} \mid \mathbf{s}\right] = \mu_3^*\left[\mathbf{d}^{23} \mid \mathbf{s}\right] = 1, \quad i = 1, 2 \tag{32}$$

$$v_1\left(\mathbf{s}\right) = \frac{1}{3\left(1-\delta\right)} - \frac{3\delta s_2}{3\left(9-\delta^2\right)} \tag{33a}$$

$$v_2\left(\mathbf{s}\right) = \frac{1}{3\left(1-\delta\right)} + \frac{\left(9-2\delta^2\right)s_2}{3\left(9-\delta^2\right)} \tag{33b}$$

$$v_3\left(\mathbf{s}\right) = \frac{1}{3\left(1-\delta\right)} - \frac{\left(3-2\delta\right)s_2}{3\left(3-\delta\right)} \tag{33c}$$

- *Case 3b:* $\mathbf{s} \in \Delta_0, s_3 \leq \frac{3\delta}{(9-\delta^2)} s_2,\ s_3 > 1 - \frac{3(6+\delta)}{(9-\delta^2)} s_2,\ s_2 \leq \frac{(9-\delta^2)}{3(6+\delta)}$

$$d_1 = d_2 = s_2, d_3 = 0 \tag{34}$$

$$\mu_i^* \left[\mathbf{d}^{3i} \mid \mathbf{s}\right] = 1, \quad i = 1,2 \tag{35}$$

$$\mu_3^* \left[\mathbf{d}^{23} \mid \mathbf{s}\right] = 1 - \mu_3^* \left[\mathbf{d}^{13} \mid \mathbf{s}\right] = \frac{1}{2} + \frac{s_1 - s_2}{s_2} \frac{\left(9 - \delta^2\right)}{2\delta\left(3 + 2\delta\right)} \tag{36}$$

$$v_1(\mathbf{s}) = \frac{1}{3(1-\delta)} - \frac{\left(\delta^2 + 3\delta - 9\right)}{6(3-\delta)\delta} s_2 - \frac{(3-\delta)}{6\delta} s_1 \tag{37a}$$

$$v_2(\mathbf{s}) = \frac{\delta}{3(1-\delta)} - \frac{\left(\delta^2 - 3\delta + 3\right)}{2(3-\delta)\delta} s_2 + \frac{(3-\delta)}{6\delta} s_1 \tag{37b}$$

$$v_3(\mathbf{s}) = \frac{\delta}{3(1-\delta)} - \frac{(3-2\delta)}{3(3-\delta)} s_2 \tag{37c}$$

- *Case 3c:* $\mathbf{s} \in \Delta_0, s_3 \leq \frac{\delta}{6+\delta},\ s_2 > \frac{(9-\delta^2)}{3(6+\delta)}$

$$d_1 = d_2 = s_2, d_3 = 0 \tag{38}$$

$$\mu_i^* \left[\mathbf{d}^{3i} \mid \mathbf{s}\right] = 1, \quad i = 1,2 \tag{39}$$

$$\mu_3^* \left[\mathbf{d}^{23} \mid \mathbf{s}\right] = 1 - \mu_3^* \left[\mathbf{d}^{13} \mid \mathbf{s}\right] = \frac{1}{2} + (s_1 - s_2) \frac{3(6+\delta)}{2\delta(3+2\delta)} \tag{40}$$

$$v_1(\mathbf{s}) = \frac{(15-\delta)}{6(1-\delta)(6+\delta)} - \frac{(s_1 - s_2)}{2\delta} \tag{41a}$$

$$v_2(\mathbf{s}) = \frac{(15-\delta)}{6(1-\delta)(6+\delta)} + \frac{(s_1 - s_2)}{2\delta} \tag{41b}$$

$$v_3(\mathbf{s}) = \frac{(3+4\delta)}{3(6 - 5\delta - \delta^2)} \tag{41c}$$

- *Case 3d:* $\mathbf{s} \in \Delta_0, s_3 > \frac{3\delta}{(9-\delta^2)} s_2,\ s_3 \leq \frac{(9-2\delta^2)}{3(3+\delta)} s_2,\ s_2 \leq \frac{27-2\delta\left(9+3\delta-\delta^2\right)}{3(18-9\delta-4\delta^2)} - \frac{3(1-\delta)(3+\delta)}{(18-9\delta-4\delta^2)} s_3$

$$d_1 = \frac{3-\delta}{3} + \frac{(3-\delta)\left(3\delta s_2 - \left(9 - 3\delta - 2\delta^2\right)(s_2 + s_3)\right)}{27 - 2\delta\left(9 + 3\delta - \delta^2\right)} \tag{42a}$$

$$d_2 = \frac{\left(9-\delta^2\right)\left((3-2\delta) s_2 - \delta s_3\right)}{27 - 2\delta\left(9 + 3\delta - \delta^2\right)} \tag{42b}$$

$$d_3 = \frac{(3-\delta)\left(\left(9-\delta^2\right) s_3 - 3\delta s_2\right)}{27 - 2\delta\left(9 + 3\delta - \delta^2\right)} \tag{42c}$$

$$\mu_1^*\left[\mathbf{d}^{31} \mid \mathbf{s}\right]=\mu_2^*\left[\mathbf{d}^{32} \mid \mathbf{s}\right]=\mu_3^*\left[\mathbf{d}^{23} \mid \mathbf{s}\right]=1 \tag{43}$$

$$v_1(\mathbf{s})=\frac{1}{3(1-\delta)}+\frac{2\delta^2 s_2-\left(9-2\delta^2\right)s_3}{27-2\delta\left(9+3\delta-\delta^2\right)} \tag{44a}$$

$$v_2(\mathbf{s})=\frac{1}{3(1-\delta)}+\frac{\left(9-2\delta^2\right)s_2-3(3+\delta)s_3}{27-2\delta\left(9+3\delta-\delta^2\right)} \tag{44b}$$

$$v_3(\mathbf{s})=\frac{1}{3(1-\delta)}+\frac{\left(18+3\delta-2\delta^2\right)s_3-9s_2}{27-2\delta\left(9+3\delta-\delta^2\right)} \tag{44c}$$

- *Case 3e:* $\mathbf{s}\in\Delta_0, s_3\leq\frac{\left(9-2\delta^2\right)}{9(3+\delta)}$, $s_2>\frac{27-2\delta\left(9+3\delta-\delta^2\right)}{3(18-9\delta-4\delta^2)}-\frac{3(1-\delta)(3+\delta)}{(18-9\delta-4\delta^2)}s_3$, $s_3\leq\frac{3\delta}{(9-\delta^2)}s_2$ *if* $s_2\leq\frac{\left(9-\delta^2\right)}{3(6+\delta)}$, $s_3>\frac{\delta}{6+\delta}$ *if* $s_2>\frac{\left(9-\delta^2\right)}{3(6+\delta)}$

$$d_1=d_2=\frac{(2\delta-3+3s_3)\left(9-\delta^2\right)}{3\left(4\delta^2+9\delta-18\right)} \tag{45a}$$

$$d_3=\frac{\delta(3-\delta)-2\left(9-\delta^2\right)s_3}{\left(4\delta^2+9\delta-18\right)} \tag{45b}$$

$$\mu_1^*\left[\mathbf{d}^{3i} \mid \mathbf{s}\right]=1, i=1,2 \tag{46}$$

$$\mu_3^*\left[\mathbf{d}^{23} \mid \mathbf{s}\right]=1-\mu_3^*\left[\mathbf{d}^{13} \mid \mathbf{s}\right]= \frac{27-\delta\left(9+6\delta+2\delta^2\right)}{\delta(3+2\delta)(3-3s_3-2\delta)}-\frac{3\left(18-9\delta-4\delta^2\right)s_2+3\left(9-3\delta-\delta^2\right)s_3}{\delta(3+2\delta)(3-3s_3-2\delta)} \tag{47}$$

$$v_1(\mathbf{s})=\frac{1}{3(1-\delta)}+\frac{(s_1-s_2)}{2\delta}-\frac{3-(15+4\delta)s_3}{2\left(4\delta^2+9\delta-18\right)} \tag{48a}$$

$$v_2(\mathbf{s})=\frac{1}{3(1-\delta)}-\frac{(s_1-s_2)}{2\delta}-\frac{3-(15+4\delta)s_3}{2\left(4\delta^2+9\delta-18\right)} \tag{48b}$$

$$v_3(\mathbf{s})=\frac{1}{3(1-\delta)}+\frac{3-(15+4\delta)s_3}{\left(4\delta^2+9\delta-18\right)} \tag{48c}$$

- *Case 3f:* $\mathbf{s}\in\Delta_0, s_3>\frac{\left(9-2\delta^2\right)}{3(3+\delta)}s_2$, $s_2\leq\frac{18+3\delta-2\delta^2}{9(3+\delta)}-s_3$

$$d_1=\frac{3-\delta}{3}-\frac{2\left(9-\delta^2\right)(s_2+s_3)}{\left(18+3\delta-2\delta^2\right)} \tag{49a}$$

$$d_2=d_3=\frac{\left(9-\delta^2\right)(s_2+s_3)}{\left(18+3\delta-2\delta^2\right)} \tag{49b}$$

$$\mu_1^*\left[\mathbf{d}^{31} \mid \mathbf{s}\right]=1-\mu_1^*\left[\mathbf{d}^{21} \mid \mathbf{s}\right]=\frac{3(3+\delta)s_2-\left(9-2\delta^2\right)s_3}{\delta(3+2\delta)(s_2+s_3)} \tag{50}$$

$$\mu_2^*\left[\mathbf{d}^{32}\mid\mathbf{s}\right]=\mu_3^*\left[\mathbf{d}^{23}\mid\mathbf{s}\right]=1 \tag{51}$$

$$v_1(\mathbf{s})=\frac{1}{3(1-\delta)}-\frac{(3+2\delta)(s_2+s_3)}{(18-3\delta+2\delta^2)} \tag{52a}$$

$$v_2(\mathbf{s})=\frac{1}{3(1-\delta)}+\frac{\left(9-2\delta^2\right)s_2-3(3+\delta)s_3}{\delta(18-3\delta+2\delta^2)} \tag{52b}$$

$$v_3(\mathbf{s})=\frac{1}{3(1-\delta)}+\frac{3(3+\delta)s_2-\left(9-2\delta^2\right)s_3}{\delta(18-3\delta+2\delta^2)} \tag{52c}$$

- *Case 3g:* $\mathbf{s}\in\Delta_0, s_2\leq\frac{1}{3},\ s_2>\frac{18+3\delta-2\delta^2}{9(3+\delta)}-s_3$

$$d_i=\frac{3-\delta}{9},\forall i \tag{53}$$

$$\mu_1^*\left[\mathbf{d}^{31}\mid\mathbf{s}\right]=1-\mu_1^*\left[\mathbf{d}^{21}\mid\mathbf{s}\right]=1-\frac{3(3+\delta)(1-3s_2)}{\delta(3+2\delta)} \tag{54}$$

$$\mu_2^*\left[\mathbf{d}^{32}\mid\mathbf{s}\right]=1-\mu_2^*\left[\mathbf{d}^{12}\mid\mathbf{s}\right]=\frac{3(3+\delta)(2-3(s_2+s_3))}{\delta(3+2\delta)} \tag{55}$$

$$\mu_3^*\left[\mathbf{d}^{23}\mid\mathbf{s}\right]=1 \tag{56}$$

$$v_i(\mathbf{s})=\frac{1}{3(1-\delta)}+\frac{1-3s_i}{3\delta},\forall i \tag{57}$$

- *Case 3h:* $\mathbf{s}\in\Delta_0, s_2>\frac{1}{3},\ s_3>\frac{\left(9-2\delta^2\right)}{9(3+\delta)}$

$$d_i=\frac{3-\delta}{9},\forall i \tag{58}$$

$$\mu_1^*\left[\mathbf{d}^{31}\mid\mathbf{s}\right]=1 \tag{59}$$

$$\mu_2^*\left[\mathbf{d}^{32}\mid\mathbf{s}\right]=1-\mu_2^*\left[\mathbf{d}^{12}\mid\mathbf{s}\right]=\frac{3(3+\delta)(1-3s_3)}{\delta(3+2\delta)} \tag{60}$$

$$\mu_3^*\left[\mathbf{d}^{23}\mid\mathbf{s}\right]=1-\mu_3^*\left[\mathbf{d}^{13}\mid\mathbf{s}\right]=1+\frac{3(3+\delta)(1-3s_2)}{\delta(3+2\delta)} \tag{61}$$

$$v_i(\mathbf{s})=\frac{1}{3(1-\delta)}+\frac{1-3s_i}{3\delta},\forall i \tag{62}$$

Proof. Assume that players play the proposal strategies in proposition 1 and that all proposals obtain a majority; then, the continuation values are as reported in proposition 1, as shown in subsection 3.i for case 1 and subsection 3.ii for cases 2a-b, and can be ascertained by direct application of equation 3 in the remainder of the cases. Then, on the basis of the definition in equation 5, we obtain player's utility functions and straightforward algebraic manipulation shows that the reported *demands* satisfy equation D1 or D2, as appropriate.

Now construct equilibrium voting strategies $A_i^*(\mathbf{s})$ $\forall i \in N$, $\mathbf{s} \in \Delta$ (by Symmetry 1) as follows:

$$A_i^*(\mathbf{s}) = \{\mathbf{x} \mid U_i(\mathbf{x}) \geq U_i(\mathbf{s})\} \tag{63}$$

These voting strategies obviously satisfy equilibrium Condition 1. Hence, it suffices to verify equilibrium Condition 2. To do so, we make use of lemmas 3 and 4. First lemma 3 establishes that the proposal strategies for legislator, say, i in Proposition 1 maximize $U_i(\mathbf{x})$ over all $\mathbf{x} \in W(\mathbf{s}) \setminus \Delta_0$; these proposals would then maximize $U_i(\mathbf{x})$ over all $\mathbf{x} \in W(\mathbf{s})$ if there is no $\mathbf{x} \in W(\mathbf{s}) \cap \Delta_0$ that accrues i higher utility, which is indeed the case by lemma 4. This completes the proof, except it remains to show that (non-degenerate) mixing probabilities in Proposition 1 are well defined in the applicable range of $\mathbf{s}$. Specifically,

- Case 2b: $\mu_3^*\left[\mathbf{d}^{23} \mid \mathbf{s}\right] \geq 0 \iff \frac{3(6+\delta)s_2-(9-\delta^2)}{\delta(3+2\delta)} \geq 0 \iff s_2 \geq \frac{(9-\delta^2)}{3(6+\delta)}$, and $\mu_3^*\left[\mathbf{d}^{23} \mid \mathbf{s}\right] \leq 1 \iff s_2 \leq \frac{1}{2} \leq \frac{9+\delta^2+3\delta}{3(6+\delta)}$.

- Case 3b: $\mu_3^*\left[\mathbf{d}^{23} \mid \mathbf{s}\right] \geq 0 \iff \frac{1}{2} + \frac{s_1-s_2}{s_2}\frac{(9-\delta^2)}{2\delta(3+2\delta)} \geq 0 \iff s_1 \geq \frac{9-3\delta-3\delta^2}{9-\delta^2}s_2 \iff s_1 \geq s_2$, $\left[\text{since } \frac{9-3\delta-3\delta^2}{9-\delta^2} \leq 1\right]$; and $\mu_3^*\left[\mathbf{d}^{23} \mid \mathbf{s}\right] \leq 1 \iff \frac{1}{2} + \frac{s_1-s_2}{s_2}\frac{(9-\delta^2)}{2\delta(3+2\delta)} \leq 1 \iff 1 - \frac{3(6+\delta)}{(9-\delta^2)}s_2 \leq s_3$.

- Case 3c: $\mu_3^*\left[\mathbf{d}^{23} \mid \mathbf{s}\right] \geq 0 \iff \frac{1}{2} + (s_1 - s_2)\frac{3(6+\delta)}{2\delta(3+2\delta)} \geq 0 \iff \frac{\delta(3+2\delta)}{3(6+\delta)} + s_1 \geq s_2$; and $\mu_3^*\left[\mathbf{d}^{23} \mid \mathbf{s}\right] \leq 1 \iff \frac{1}{2} + (s_1 - s_2)\frac{3(6+\delta)}{2\delta(3+2\delta)} \leq 1 \iff s_1 - \frac{\delta(3+2\delta)}{3(6+\delta)} \leq s_2 \iff 1 - \frac{(9-\delta^2)}{3(6+\delta)} - \frac{\delta(3+2\delta)}{3(6+\delta)} \leq s_2$, $\left[\text{since } \sup s_1 = 1 - \frac{(9-\delta^2)}{3(6+\delta)}\right] \iff \frac{(9-\delta^2)}{3(6+\delta)} \leq s_2$.

- Case 3e: $\mu_3^*\left[\mathbf{d}^{23} \mid \mathbf{s}\right] \geq 0 \iff \frac{27-\delta(9+6\delta+2\delta^2)}{\delta(3+2\delta)(3-3s_3-2\delta)} - \frac{3(18-9\delta-4\delta^2)s_2+3(9-3\delta-\delta^2)s_3}{\delta(3+2\delta)(3-3s_3-2\delta)} \geq 0 \iff \frac{27-\delta(9+6\delta+2\delta^2)}{3(18-9\delta-4\delta^2)} - \frac{(9-3\delta-\delta^2)}{(18-9\delta-4\delta^2)}s_3 \geq s_2 \iff \frac{27-\delta(9+6\delta+2\delta^2)}{3(18-9\delta-4\delta^2)} - \frac{(9-3\delta-\delta^2)}{(18-9\delta-4\delta^2)}s_3 \geq \frac{1-s_3}{2}$, $\left[\text{since } \max s_2 = \frac{1-s_3}{2}\right] \cdot s_3 \leq 1-\frac{2}{3}\delta$. Also $\mu_3^*\left[\mathbf{d}^{23} \mid \mathbf{s}\right] \leq 1 \iff \frac{27-\delta(9+6\delta+2\delta^2)}{\delta(3+2\delta)(3-3s_3-2\delta)} - \frac{3(18-9\delta-4\delta^2)s_2+3(9-3\delta-\delta^2)s_3}{\delta(3+2\delta)(3-3s_3-2\delta)} \leq 1 \iff \frac{27-2\delta(9+3\delta-\delta^2)}{3(18-9\delta-4\delta^2)} - \frac{3(1-\delta)(3+\delta)}{(18-9\delta-4\delta^2)}s_3 \leq s_2$.

- Case 3f : $\mu_1^*\left[\mathbf{d}^{31} \mid \mathbf{s}\right] \geq 0 \iff \frac{3(3+\delta)s_2-(9-2\delta^2)s_3}{\delta(3+2\delta)(s_2+s_3)} \geq 0 \iff s_2 \geq \frac{(9-2\delta^2)}{3(3+\delta)}s_3$; and $\mu_1^*\left[\mathbf{d}^{31} \mid \mathbf{s}\right] \leq 1 \iff \frac{3(3+\delta)s_2-(9-2\delta^2)s_3}{\delta(3+2\delta)(s_2+s_3)} \leq 1 \iff s_3 \geq \frac{(9-2\delta^2)}{3(3+\delta)}s_2$.

- Case 3g: $\mu_1^*\left[\mathbf{d}^{31}\mid\mathbf{s}\right]\geq 0\Longleftrightarrow 1-\frac{3(3+\delta)(1-3s_2)}{\delta(3+2\delta)}\geq 0\Longleftrightarrow s_2\geq\frac{(9-2\delta^2)}{9(3+\delta)}$; and $\mu_1^*\left[\mathbf{d}^{31}\mid\mathbf{s}\right]\leq 1\Longleftrightarrow 1-\frac{3(3+\delta)(1-3s_2)}{\delta(3+2\delta)}\leq 1\Longleftrightarrow s_2\leq\frac{1}{3}$. Also, $\mu_2^*\left[\mathbf{d}^{32}\mid\mathbf{s}\right]\geq 0\Longleftrightarrow\frac{3(3+\delta)(2-3(s_2+s_3))}{\delta(3+2\delta)}\geq 0\Longleftrightarrow\frac{2}{3}\geq s_2+s_3$; and $\mu_2^*\left[\mathbf{d}^{32}\mid\mathbf{s}\right]\leq 1\Longleftrightarrow\frac{3(3+\delta)(2-3(s_2+s_3))}{\delta(3+2\delta)}\leq 1\Longleftrightarrow\frac{18+3\delta-2\delta^2}{9(3+\delta)}-s_3\leq s_2$.

- Case 3h: $\mu_2^*\left[\mathbf{d}^{32}\mid\mathbf{s}\right]\geq 0\Longleftrightarrow\frac{3(3+\delta)(1-3s_3)}{\delta(3+2\delta)}\geq 0\Longleftrightarrow s_3\leq\frac{1}{3}$ and $\mu_2^*\left[\mathbf{d}^{32}\mid\mathbf{s}\right]\leq 1\Longleftrightarrow\frac{3(3+\delta)(1-3s_3)}{\delta(3+2\delta)}\leq 1\Longleftrightarrow\frac{9-2\delta^2}{9(3+\delta)}\leq s_3$. Also $\mu_3^*\left[\mathbf{d}^{23}\mid\mathbf{s}\right]\geq 0\Longleftrightarrow 1+\frac{3(3+\delta)(1-3s_2)}{\delta(3+2\delta)}\geq 0\Longleftrightarrow s_2\geq\frac{9-2\delta^2}{9(3+\delta)}$ and $\mu_3^*\left[\mathbf{d}^{23}\mid\mathbf{s}\right]\leq 1\Longleftrightarrow\frac{3(3+\delta)(1-3s_2)}{\delta(3+2\delta)}\leq 0\Longleftrightarrow s_2\geq\frac{1}{3}$.

■

The following lemmas apply for the demands, value functions, and equilibrium proposal strategies reported in Proposition 1:

Lemma 1 $\forall\ (x,1-x,0)\in\Delta$, *(a)* $U_i(x,1-x,0), i=1,2$ *is continuous and piece-wise differentiable with respect to* x*, and (b)* $\frac{\partial U_1(x,1-x,0)}{\partial x}\geq 0$.

Proof. We have

$$U_1(x,1-x,0)=\begin{cases}1+\frac{\delta}{3(1-\delta)} & if \quad x=1\\ x+\delta\left[\frac{1}{3(1-\delta)}-\frac{\delta(1-x)}{(9-\delta^2)}\right] & if \quad x\in\left[1-\frac{(9-\delta^2)}{3(6+\delta)},1\right)\\ \frac{1}{2}+\frac{\delta(15-\delta)}{6(1-\delta)(6+\delta)} & if \quad x\in\left(\frac{(9-\delta^2)}{3(6+\delta)},1-\frac{(9-\delta^2)}{3(6+\delta)}\right)\\ x+\delta\left[\frac{1}{3(1-\delta)}+\frac{x}{(3-\delta)}\right] & if \quad x\in\left(0,\frac{(9-\delta^2)}{3(6+\delta)}\right]\\ \frac{\delta}{3(1-\delta)} & if \quad x=0\end{cases}\tag{64}$$

hence $\lim_{x\longrightarrow 1}\left[x+\delta\left[\frac{1}{3(1-\delta)}-\frac{\delta(1-x)}{(9-\delta^2)}\right]\right]=1+\frac{\delta}{3(1-\delta)}$, $\lim_{x\longrightarrow 0}\left[x+\delta\left[\frac{1}{3(1-\delta)}+\frac{x}{(3-\delta)}\right]\right]=\frac{\delta}{3(1-\delta)}$, and $U_1\left(1-\frac{(9-\delta^2)}{3(6+\delta)},\frac{(9-\delta^2)}{3(6+\delta)},0\right)=U_1\left(\frac{(9-\delta^2)}{3(6+\delta)},1-\frac{(9-\delta^2)}{3(6+\delta)},0\right)=\frac{1}{2}+\frac{\delta(15-\delta)}{6(1-\delta)(6+\delta)}$, which proves (a) for $i=1,2$ by symmetry. For (b), we have $\frac{\partial U_1(x,1-x,0)}{\partial x}=1+\frac{\delta^2}{(9-\delta^2)}>0$ for $x\in\left[1-\frac{(9-\delta^2)}{3(6+\delta)},1\right)$, $\frac{\partial U_1(x,1-x,0)}{\partial x}=0$ for $x\in\left(\frac{(9-\delta^2)}{3(6+\delta)},1-\frac{(9-\delta^2)}{3(6+\delta)}\right)$, and $\frac{\partial U_1(x,1-x,0)}{\partial x}=1+\frac{\delta}{(3-\delta)}>0$ for $x\in\left(0,\frac{(9-\delta^2)}{3(6+\delta)}\right]$. ■

Lemma 2 $\forall\mathbf{s}\in\Delta$, *(a)* $\sum_{i=1}^{3}d_i(\mathbf{s})\leq 1$*, and (b)* $s_i\geq s_j\Longrightarrow d_i(\mathbf{s})\geq d_j(\mathbf{s})$.

Proof. By symmetry it suffices to consider $\mathbf{s}$ such that $s_1 \geq s_2 \geq s_3$, whence part (b) reduces to showing $d_1 \geq d_2 \geq d_3$. We have the following cases:

- Case 1: $\sum_{i=1}^{3} d_i(\mathbf{s}) = 1$, for (a) and $1 = d_1 \geq d_2 = d_3 = 0$ for (b).
- Cases 2a,3a: $\sum_{i=1}^{3} d_i(\mathbf{s}) = \frac{(9-\delta^2)-9s_2}{3(3+\delta)} + s_2 \leq 1 \iff s_2 \leq 1 + \frac{\delta}{3}$, for (a). For (b), $d_2 = s_2 \geq d_3 = 0$ and $d_1 \geq d_2 \iff \frac{(9-\delta^2)-9s_2}{3(3+\delta)} \geq s_2 \iff s_2 \leq \frac{(9-\delta^2)}{3(6+\delta)}$.
- Cases 2b,3b-c: $\sum_{i=1}^{3} d_i(\mathbf{s}) = 2s_2 \leq 1 \iff s_2 \leq \frac{1}{2}$, for (a). For (b), $d_1 = d_2 = s_2 \geq d_3 = 0$
- Case 3d: $\sum_{i=1}^{3} d_i(\mathbf{s}) = \frac{3-\delta}{3} + \frac{(3-\delta)(3\delta s_2-(9-3\delta-2\delta^2)(s_2+s_3))}{27-2\delta(9+3\delta-\delta^2)} + \frac{(9-\delta^2)((3-2\delta)s_2-\delta s_3)}{27-2\delta(9+3\delta-\delta^2)} + \frac{(3-\delta)((9-\delta^2)s_3-3\delta s_2)}{27-2\delta(9+3\delta-\delta^2)} = 1 - \frac{\delta}{3} \leq 1 \implies \delta \geq 0$, for (a). For (b) we have $d_1 \geq d_2 \iff \frac{3-\delta}{3} + \frac{(3-\delta)(3\delta s_2-(9-3\delta-2\delta^2)(s_2+s_3))}{27-2\delta(9+3\delta-\delta^2)} \geq \frac{(9-\delta^2)((3-2\delta)s_2-\delta s_3)}{27-2\delta(9+3\delta-\delta^2)} \iff s_2 \leq \frac{27-2\delta(9+3\delta-\delta^2)}{3(18-9\delta-4\delta^2)} - \frac{3(1-\delta)(3+\delta)}{(18-9\delta-4\delta^2)} s_3$, and $d_2 \geq d_3 \iff \frac{(9-\delta^2)((3-2\delta)s_2-\delta s_3)}{27-2\delta(9+3\delta-\delta^2)} \geq \frac{(3-\delta)((9-\delta^2)s_3-3\delta s_2)}{27-2\delta(9+3\delta-\delta^2)} \iff s_3 \leq \frac{(9-2\delta^2)}{3(3+\delta)} s_2$.
- Case 3e: $\sum_{i=1}^{3} d_i(\mathbf{s}) = 2\frac{(2\delta-3+3s_3)(9-\delta^2)}{3(4\delta^2+9\delta-18)} + \frac{\delta(3-\delta)-2(9-\delta^2)s_3}{(4\delta^2+9\delta-18)} = 1 - \frac{\delta}{3} \leq 1 \implies \delta \geq 0$, for (a). For (b) $d_1 = d_2 \geq d_3 \iff \frac{(2\delta-3+3s_3)(9-\delta^2)}{3(4\delta^2+9\delta-18)} \geq \frac{\delta(3-\delta)-2(9-\delta^2)s_3}{(4\delta^2+9\delta-18)} \iff s_3 \leq \frac{9-2\delta^2}{9(3+\delta)}$.
- Case 3f: $\sum_{i=1}^{3} d_i(\mathbf{s}) = \frac{3-\delta}{3} - \frac{2(9-\delta^2)(s_2+s_3)}{(18+3\delta-2\delta^2)} + 2\frac{(9-\delta^2)(s_2+s_3)}{(18+3\delta-2\delta^2)} = 1 - \frac{\delta}{3} \leq 1 \implies \delta \geq 0$ for (a). For (b) $d_1 \geq d_2 = d_3 \iff \frac{3-\delta}{3} - \frac{2(9-\delta^2)(s_2+s_3)}{(18+3\delta-2\delta^2)} \geq \frac{(9-\delta^2)(s_2+s_3)}{(18+3\delta-2\delta^2)} \iff \frac{(18+3\delta-2\delta^2)}{9(3+\delta)} - s_3 \geq s_2$.
- Cases 3g-h: $\sum_{i=1}^{3} d_i(\mathbf{s}) = 3\frac{3-\delta}{9} = 1 - \frac{\delta}{3} \leq 1 \implies \delta \geq 0$, for (a) and (b) holds trivially.

■

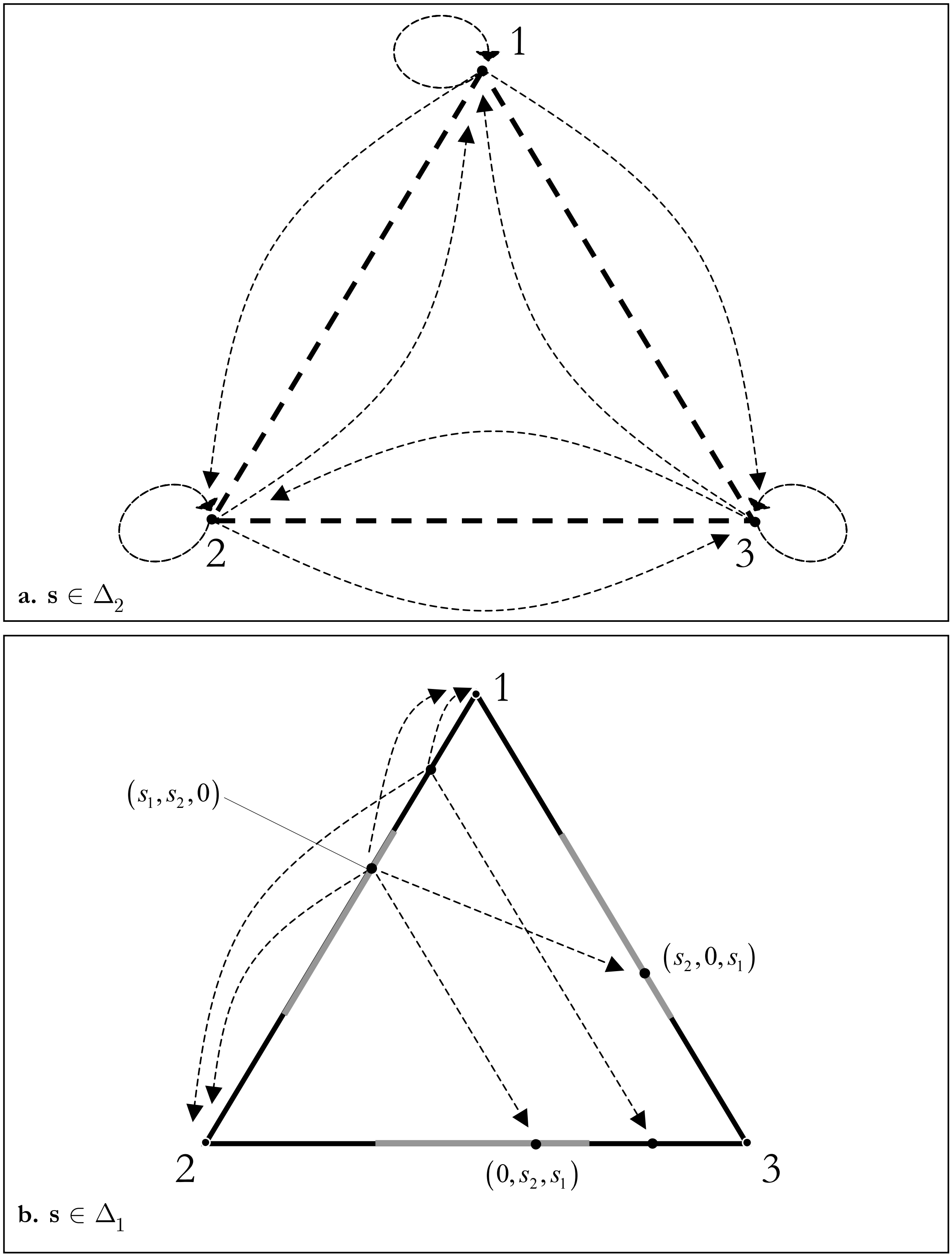
Lemma 3 $\mu_i[\mathbf{z} \mid \mathbf{s}] > 0 \implies \mathbf{z} \in \arg\max\{U_i(\mathbf{x}) \mid \mathbf{x} \in W(\mathbf{s}) \setminus \Delta_0\}$, $\forall \mathbf{z}, \mathbf{s} \in \Delta$.

Proof. All equilibrium proposals take the form of minimum-winning consistent proposals $\mathbf{d}^{ji} \in \Delta \setminus \Delta_0$. By lemma 1, $\mathbf{z} = \mathbf{d}^{ji} \implies \mathbf{z} \in \arg\max\left\{U_i(\mathbf{x}) \mid \mathbf{x} \in A_i^*(\mathbf{s}) \cap A_j^*(\mathbf{s})\right\}$, where $\mathbf{d}^{ji} \in A_i^*(\mathbf{s})$ is guaranteed by part (a) of lemma 2. Also note that in all cases of proposition 1 when $\mu_i\left[\mathbf{d}^{ji} \mid \mathbf{s}\right] > 0$ and $\mu_i\left[\mathbf{d}^{hi} \mid \mathbf{s}\right] > 0, h \neq j$ it is also the case that $d_h = d_j$, so that $U_i\left(\mathbf{d}^{ji}\right) = U_i\left(\mathbf{d}^{hi}\right)$ by symmetry. Then it suffices to show that $\mu_i\left[\mathbf{d}^{ji} \mid \mathbf{s}\right] = 0 \Longrightarrow U_i\left(\mathbf{d}^{ji}\right) \leq U_i\left(\mathbf{d}^{hi}\right) \Longleftrightarrow d_h \leq d_j, h \neq j$ the latter equivalence from lemma 1. Indeed, for $\mu_i\left[\mathbf{d}^{ji} \mid \mathbf{s}\right] = 0 = 1 - \mu_i\left[\mathbf{d}^{hi} \mid \mathbf{s}\right] \Longrightarrow j > h$ in Proposition 1, which by part (b) of lemma 2 completes the proof. ■

Lemma 4 $\forall \mathbf{x} \in (W(\mathbf{s}) \cap \Delta_0), \exists \mathbf{y} \in W(\mathbf{s}) \setminus \Delta_0 \ni U_i(\mathbf{y}) \geq U_i(\mathbf{x}), \forall i \in N.$

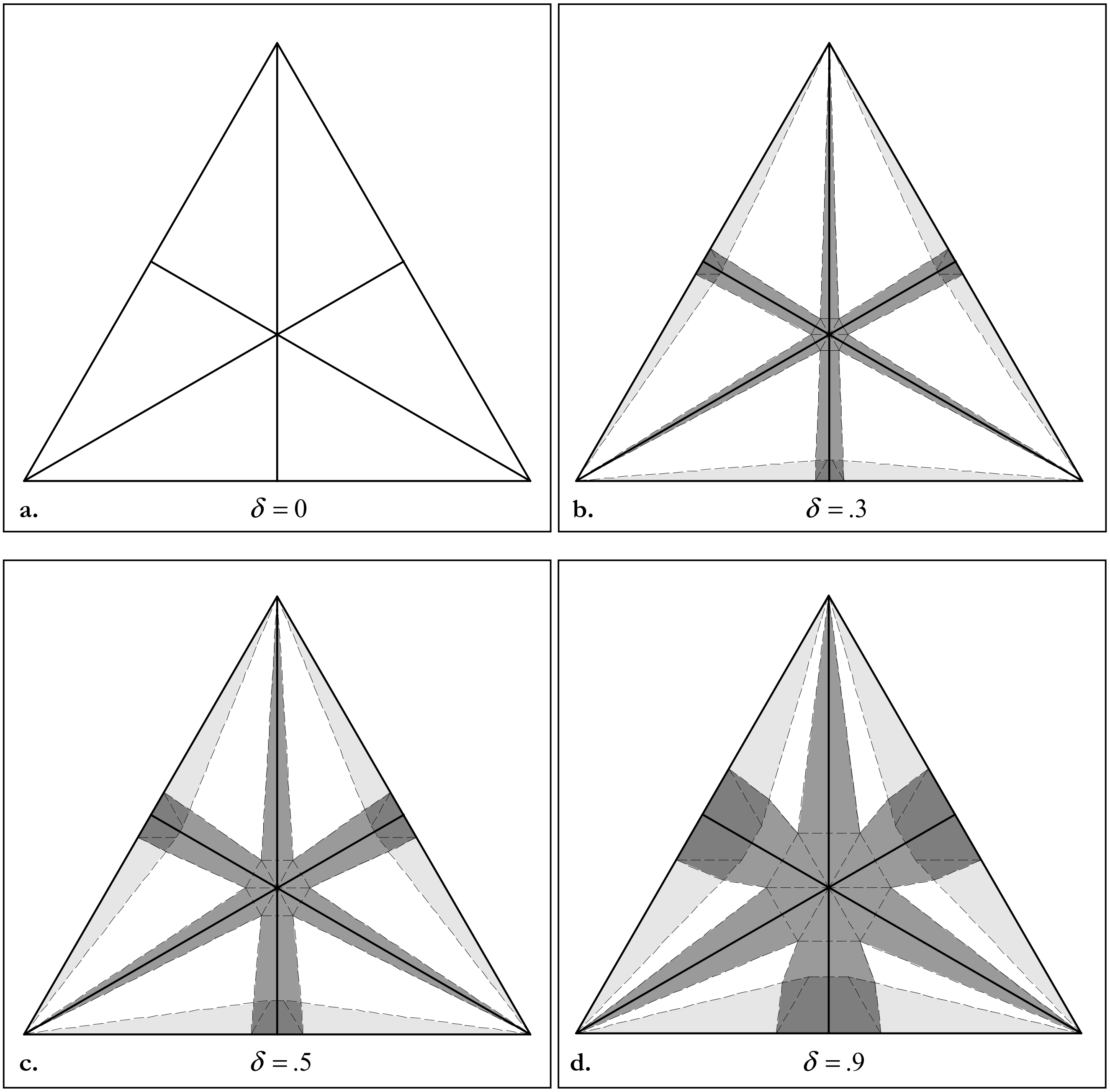
Proof. Since $\mathbf{s} \in W(\mathbf{s})$ it suffices to consider $\mathbf{x} \in (W(\mathbf{s}) \cap \Delta_0 \cap A_i^*(\mathbf{s}))$ [if $U_i(\mathbf{x}) < U_i(\mathbf{s})$ we can apply the argument that follows for $\mathbf{x} = \mathbf{s}$]. Then, without loss of generality, let $\mathbf{x}$ be preferred to $\mathbf{s}$ by a majority of (at least) i and j. Now set $\mathbf{y} = \mathbf{d}^{ji}(\mathbf{x})$; by definition $U_j\left(\mathbf{d}^{ji}(\mathbf{x})\right) \geq U_j(\mathbf{x})$. It is also the case that $U_i\left(\mathbf{d}^{ji}(\mathbf{x})\right) \geq U_i(\mathbf{x})$; this follows by part (b) of Lemma 1 and the fact that $d_i^{ji}(\mathbf{x}) = 1 - d_j(\mathbf{x}) \geq d_i(\mathbf{x}) \Longleftrightarrow 1 \geq d_i(\mathbf{x}) + d_j(\mathbf{x})$, where the latter is true by part (a) of Lemma 2. But then transitivity of individual preferences ensures that $\mathbf{y} = \mathbf{d}^{ji}(\mathbf{x}) \in W(\mathbf{s})$ by a majority of (at least) i and j. ■

Figure 1: Equilibrium Induced Markov Process -- $s \in \Delta_{1,2}$



Key: a. Δ_2 is an irreducible absorbing set; **b.** player 3 mixes when difference between s_1 and s_2 is small (allocations marked with ▬▬).
Sources: Constructed by author on basis of Proposition 1.

Figure 2: Demands and Equilibrium Proposal Strategies vs. Discount Factor -- $s \in \Delta_0$

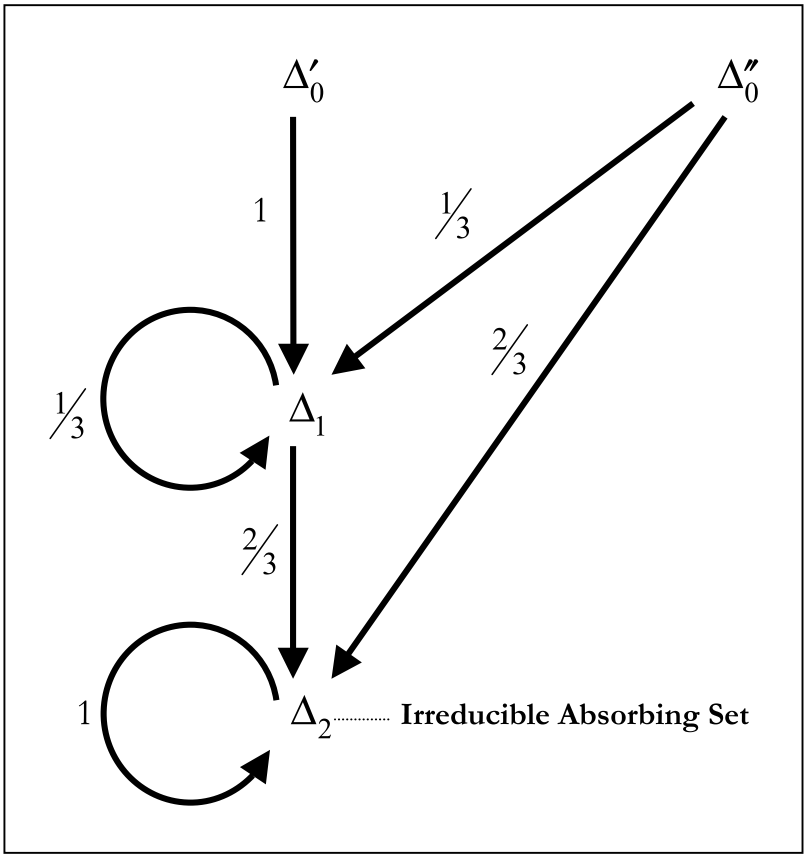


Key: Allocations for which mixed proposal strategies are played and/or legislators demand zero, expand with larger discount factors.

- Legislator with min. amount demands zero & plays mixed proposal strategy (Prop. 1, cases 3b-c).
- Proposer(s) play mixed proposal strategies (Prop. 1, cases 3e-h).
- Legislator with min. amount demands zero (Prop. 1, case 3a).
- All legislators demand positive amount. Proposers play pure strategies (Prop. 1, case 3d).

Sources: Constructed by author on basis of Proposition 1.

Figure 3: Equilibrium Induced Markov Process



Key: Δ_0', cases 3d-h of Proposition 1. Δ_0'', cases 3a-c of Proposition 1. Δ_1, cases 2a-b of Proposition 1. Δ_2, case 1 of Proposition 1.
Sources: Constructed by author on basis of Proposition 1.