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Information, heterogeneity and market incompleteness

by Liam Graham and Stephen Wright

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JEL classification: D52, D84, E32

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Information, heterogeneity and market incompleteness[*]

Liam Graham[†] and Stephen Wright[‡]

31 October 2008

Abstract

We provide a microfounded account of imperfect information in a dynamic general equilibrium model by describing heterogeneous households that acquire information only through their participation in markets. Thus incomplete markets will imply incomplete information. We solve the model taking full account of the infinite regress of expectations, and show that the properties of the model change dramatically. Under virtually all calibrations the impact response of consumption to a positive aggregate technology shock is negative. If households observe a noisy public signal in addition to the information they obtain from markets, consumption responds to shocks sluggishly.

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1 Introduction

Most dynamic general equilibrium models assume that households can perfectly observe the state variables. Complete markets rationalize this assumption: in a decentralized equilibrium households learn about aggregates through participating in markets, so if markets are complete so too will be information.[1] However if markets are incomplete, households will in general be imperfectly informed about the aggregate economy, and hence about other agents.

To investigate these issues, we describe a simple dynamic general equilibrium model of a type which is becoming standard in work on incomplete markets and heterogeneity[2]: households are heterogenous because they face an idiosyncratic productivity shock in addition to an aggregate productivity shock.

What is not standard is the link we make between market structure and information. Households can only observe their own allocations, along with the prices in the limited set of markets in which they participate. This results in heterogeneous and incomplete household information sets. In equilibrium households have to solve a filtering problem knowing that all other households are solving a symmetric problem, but on the basis of different information sets.

We make four main contributions to the dynamic general equilibrium literature:

1. We provide analytical and numerical results to show that incomplete markets can dramatically change the dynamics of DGE models when their informational implications are considered.

2. If we add a noisy public signal (such as a statistical office's measure of output) the response of consumption becomes hump-shaped. In contrast to much of the literature[3], we obtain this result in a model

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[1] This equivalence goes at least as far back as Grossman and Stiglitz (1980)

[2] Examples of papers which use similar models include Krusell and Smith (1998); Nakajima (2005); Lorenzoni (2006); Porapakkarm and Young (2008)

[3] For example Smets and Wouters (2007).

without real or nominal rigidities.

3. Our results are robust to the addition of aggregate financial assets
4. In modelling terms, we show how a rational expectations equilibrium can be derived in a DSGE with heterogeneous information sets and a fully consistent treatment of the resulting hierarchy of expectations (for which we adopt techniques developed by Nimark, 2007, 2008).

In the standard stochastic growth model (which is a complete-markets version of our economy) the impact effect of a positive aggregate productivity shock on consumption is positive[4]. The same is true if markets are incomplete but full information is simply assumed. However, we find that, with incomplete markets and a consistent treatment of information the impact response of consumption becomes negative under a wide range of calibrations, and the subsequent path of aggregate consumption is very different from the full information case.

The intuition for this is as follows. A household observes a positive innovation to aggregate productivity only indirectly, via positive innovations in its wage and the return on capital. We show analytically that, as long as the variance of the idiosyncratic shock is non-zero, full information cannot be recovered either instantaneously or asymptotically. The strong empirical evidence (for example Guvenen, 2005, 2007) that the variance of idiosyncratic productivity shocks is much higher than that of aggregate shocks means that the wage contains little useful information about aggregates, so the main signal the household receives is a positive innovation to the return to capital. Households know that such a positive innovation to returns could be caused either by a positive innovation to aggregate productivity or by aggregate capital being lower than the household had previously estimated. We show analytically that the second effect will, under reasonable parameter restrictions, always cause the consumption response to be less than under full

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[4] Campbell (1994) gives precise conditions under which the impact response of consumption is positive. A sufficient condition is that the coefficient of relative risk aversion is greater than or equal to unity.

information; and we show numerically that under a wide range of calibrations the impact response of consumption to the shock is actually negative.

We then perform two robustness checks. Firstly, we introduce a noisy public signal on aggregates and find that the impact response of consumption is close to zero, and is subsequently hump-shaped: imperfect information greatly strengthens the model's endogenous propagation mechanisms. Secondly, we introduce a risk-free bond and a stock market. We find that such "aggregate" assets do not add much to the information already at households' disposal, and so do not greatly affect our results.

Our results are in contrast to much of the macro literature on incomplete markets and heterogeneity[5], which makes extensive use of Krusell and Smith's (1998) result that the distribution of wealth arising from incomplete markets has only a small impact on aggregate dynamics. As a result "approximate aggregation" is possible and households can accurately forecast using a law of motion expressed as a function of aggregate capital and aggregate technology. However, this approach relies on the assumption of full information without any account of where this information comes from.

The uncertainty about the true values of the states that arises in our model seems plausible, particularly given recent debates about the true size of the capital stock[6]. However, a quite striking feature of our results is that the actual uncertainty in households' estimates in our simple model is quite small. This suggests we may be considerably understating the informational problem households face.

The paper proceeds as follows. Section 2 gives a brief review of the existing literature then section 3 describes the model and section 4 considers two benchmark cases with full information. In section 5, we formalize the information set of agents, show how the infinite hierarchy of expectations arises and define the equilibrium. Section 6 presents our analytical results, and section 7 gives numerical results along with robustness checks and a discussion of empirical implications. Section 8 concludes.

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[5] For a review see Heathcote et al (2008).

[6] For example, Hall (2001) or the discussion of intangibles in Laitner and Stolyarov (2003).

2 Relation to the literature

The study of imperfect information has a long history in macroeconomics (Hayek, 1945, Lucas, 1972). Here we focus on the most relevant part of the literature, in which the private sector is imperfectly informed[7]. One strand of this literature[8] investigates the implications of noisy indicators in a representative agent framework. However all these papers suffer from the limitation that the existence of a representative agent requires either complete markets or a continuum of identical agents, and in both of these cases we show that full information is trivially revealed. Another strand focusses on the roles of limited information and limited informational processing in propagating shocks[9]. A more complete review can be found in Hellwig (2006).

Two recent papers present models that are more similar to ours, with incomplete markets and households facing both an aggregate and an idiosyncratic productivity shock. Lorenzoni (2006) presents such a model, with the addition of two noise shocks (a "sampling shock" and a shock to the rate of return) and noise in a public signal of aggregate productivity, and shows that this gives a new explanation for the existence of demand shocks. While valuable insights are to be gained from such a study of the effects of noisy indicators, one important contribution of our paper is to show that such noise is not necessary to motivate informational problems.

Porapakkarm and Young's (2008) underlying model is also close to ours, but they make strong assumptions to reduce the impact of informational heterogeneity. Specifically, they assume that partially informed agents ignore the additional dynamics due to higher order expectations when they make their savings decisions and believe that the law of motion of capital is captured by an equation which only includes aggregate capital.

In contrast, our analytical results involve no such simplifying assumptions. Households in our model, as in the two papers cited above, will in gen-

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[7] There is also a related literature which considers the problem of a monetary policymaker with imperfect information (Pearlman et al, 1986; Pearlman, 1992; Svensson and Woodford, 2002, 2004, Aoki, 2003, 2006).

[8] Bomfim (2001), Keen (2004), Collard and Dellas (2006).

[9] Hellwig (2005), Barsky and Sims (2006), Adam (2006), Bacchetta and van Wincoop (2006), Mackowiak and Wiederholt (2006), Veldkamp and Wolfers (2007).

eral have idiosyncratic information sets and so, to forecast aggregate states will need to form expectations of other households' expectations. Townsend (1983) first analysed the problem of "forecasting the forecasts of others" and the infinite regress of expectations that results. Woodford (2003) shows that the dynamics of such higher-order expectations can lead to shocks having more persistent effects. Here we draw on recent work by Nimark (2007) who derives new techniques for modelling the resulting infinite-dimensional state vector when agents make dynamic choices[10].

3 The stochastic growth model with idiosyncratic productivity shocks

In this section we present a model of the type that is becoming standard in the dynamic general equilibrium literature[11]. Our economy consists of a large number of households and a large number of firms, divided into S islands, on each of which there a unit mass of firms and households. Households consume, rent capital and labour to firms and are subject to an island-specific shock to their labour productivity. Firms use capital and labour to produce a single consumption good with a technology that is subject to an aggregate productivity shock. Markets are incomplete in that the only asset available is capital,[12] and while labour is heterogenous across islands, capital is homogenous and can freely flow between islands. We focus here on the key structural relationships; the full log-linearised model is provided in Appendix A.

We use upper case letters for levels, lower case letters for log deviations from the steady state growth path. A superscript s indicates a variable relating to a typical household or firm on island s. Without the superscript

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[10] Nimark's (2007b) analysis of the inference problem with higher expectations in a model of sticky prices is also a rare example of a paper in which, as in ours, imperfect information is due to heterogeneity rather than arbitrary noise.

[11] Examples of papers which use similar models include Krussel and Smith (1998); Nakajima (2005); Lorenzoni (2006); Porapakkarm and Young (2008)

[12] We relax this assumption in Section 7.7, where we show that introducing a risk-free asset and a stock market has little impact on our results.

the variable is an aggregate.

3.1 Households

A typical household on island s consumes (C_t^s) and rents capital (K_t^s) and labour (H_t^s) to firms. Household labour on each island has idiosyncratic productivity (Z_t^s) whereas capital is homogenous, so households earn the aggregate return (R_{kt}) on capital but an idiosyncratic wage (V_t^s) on their labour. Apart from the idiosyncratic shock, households on different islands are identical.

The problem of a household on island s is to choose paths for consumption, labour supply and investment (I_t^s) to maximize expected lifetime utility given by

$$E_t^s \sum_{i=0}^{\infty} \beta^i \left[\ln C_{t+i}^s + \theta \frac{\left(1 - H_{t+i}^s\right)^{1-\gamma}}{1-\gamma} \right] \tag{1}$$

where $\frac{1}{\gamma}$ is the intertemporal elasticity of labour supply, and β the subjective discount rate, subject to a resource constraint

$$R_{kt} K_t^s + V_t^s H_t^s = C_t^s + I_t^s \tag{2}$$

and the evolution of the household's holdings of capital

$$K_{t+1}^s = (1-\delta) K_t^s + I_t^s \tag{3}$$

The expectations operator for an individual household is defined as the expectation given the household's information set Ω_t^s, i.e. for some variable a_t

$$E_t^s a_t = E a_t | \Omega_t^s \tag{4}$$

The household's first-order conditions consist of an Euler equation

$$\frac{1}{C_t^s} = \beta E_t^s \left[\frac{R_{t+1}}{C_{t+1}^s} \right] \tag{5}$$

where $R_t = R_{kt} + 1 - \delta$ is the gross return to a one-period investment in capital, and a labour supply relation

$$\theta\left(1-H_t^s\right)^{-\gamma}=\frac{V_t^s}{C_t^s} \tag{6}$$

3.2 Firms

The production function of a typical firm on island s is

$$Y_t^s=\left(J_t^s\right)^{1-\alpha}\left(A_t Z_t^s H_t^s\right)^a \tag{7}$$

where A_t is an aggregate productivity shock and J_t^s is the capital rented by the firm: in general, $J_t^s \neq K_t^s$, since capital will flow to more productive islands.

The first-order conditions of this firm are

$$R_{kt}=\frac{Y_t^s}{J_t^s} \tag{8}$$

$$V_t^s=\frac{Y_t^s}{H_t^s} \tag{9}$$

3.3 Aggregates

Aggregate quantities are sums over household or firm quantities, and for convenience we calculate them as quantities per household. For example aggregate consumption is given by

$$C_t=\frac{1}{S}\sum_{s=1}^{S} C_t^s \tag{10}$$

The economy's aggregate resource constraint is then

$$Y_t=C_t+I_t \tag{11}$$

3.4 Markets

Labour markets are completely segmented between islands, so firms on island s only rent labour from households on island s, and the wage on island s, V_t^s, adjusts to set labour supply (6) equal to labour demand (9).

In contrast, capital is homogenous and tradeable between islands, so flows to islands with more productive labour. The aggregate return to capital, R_t, adjusts to clear the capital market, making the demand for capital for each firm (8) consistent with each household's Euler equation (5) and the aggregate resource constraint (11).

3.5 Shocks

For both the aggregate and idiosyncratic productivity shocks we assume autoregressive processes in log deviations

$$\begin{aligned} a_t &= \phi_a a_{t-1} + \varpi_t \quad (12) \\ z_t^s &= \phi_z z_{t-1}^s + \varpi_t^s \quad (13) \end{aligned}$$

where ϖ_t and ϖ_t^s are i.i.d mean-zero errors, and $E\varpi_t^2 = \sigma_a^2$; $E\left(\varpi_t^s\right)^2 = \sigma_z^2$. The innovation to the idiosyncratic process satisfies an adding up constraint, $\sum\limits_{s=1}^{S} \varpi_t^s = 0$ which implies

$$\sum_{s=1}^{S} z_t^s = 0. \quad (14)$$

3.6 Equilibrium

While our underlying model is non-linear, we work with the log-linear approximation to the model which we require in order to be able to use a linear filter to model the household's signal extraction problem[13].

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[13] Although non-linear filters exist, incorporating one in this type of model and dealing with the hierarchy of expectations in a non-linear world would present a formidable challenge.

We show in Appendix A that the features of the economy relevant to a household on island s can be written as an Euler equation

$$E_t^s \Delta c_{t+1}^s = E_t^s r_{t+1} \tag{15}$$

and a linearised law of motion for the economy that is symmetric across islands:

$$W_{t+1}^s = F_W W_t^s + F_c c_t + F_s c_t^s + u_t^s \tag{16}$$

where $W_t^s = \left[\begin{array}{cc} \xi_t' & \chi_t^{s\prime} \end{array} \right]'$ is a vector of underlying states relevant to a household on island s comprising aggregate states $\xi_t = \left[\begin{array}{cc} k_t & a_t \end{array} \right]'$ and states specific to the household, given by $\chi_t^s = \left[\begin{array}{cc} k_t^s - k_t & z_t^s \end{array} \right]'$. The innovation vector for W_t is $u_t^s = \left[\begin{array}{cccc} 0 & \varpi_t & 0 & \varpi_t^s \end{array} \right]'$ (since both k_t and k_t^s are pre-determined) The coefficient matrices F_W, F_c and F_s are defined in Appendix A.4.

The linearisation is very close to that Campbell (1994): indeed the coefficients for the aggregate part of our economy are identical to his.

Definition 1 ***(Equilibrium)*** *A competitive equilibrium for the above economy is a sequence of plans for*

- allocations of households $\left\{c_t^s, h_t^s, k_{t+1}^s\right\}_{t=1:\infty}^{s=1:S}$
- prices $\{r_t, v_t^s\}_{t=1:\infty}^{s=1:S}$
- aggregate factor inputs $\{k_t, h_t\}_{t=1:\infty}$

such that

1. Given prices and informational restrictions, the allocations solve the utility maximization problem for each consumer
2. $\{r_t, v_t^s\}_{t=1:\infty}^{s=1:S}$ are the marginal products of aggregate capital and island-specific labour.
3. All markets clear

4 Two benchmark cases

The key assumption of this paper is that the only tradeable asset is capital, and, consistent with a decentralized equilibrium, agents are not directly provided with information on the aggregate states. However, as benchmark cases we first investigate the case of complete markets, which we show reveal full information, and, second, that of incomplete markets with full information simply assumed.

Definition 2 ***(Full information)*** *Full information, which we denote by an information set Ω_t^*, is knowledge of the aggregate states in the economy ξ_t, the idiosyncratic states χ_t^s of all households and the time-invariant parameters and structure of the underlying model Ξ.*

$$\Omega_t^* = \left[\xi_t, \{\chi_t^s\}_{s=1}^S, \Xi \right] \tag{17}$$

4.1 Complete markets

Complete markets imply the existence of a set of securities that span the distribution of idiosyncratic shocks. Thus complete risk-sharing is possible[14] and in the process the household productivity shocks z^s are revealed to all households, so each household knows both the aggregate wage and the full set of disaggregate wages. Risk-sharing implies that household paths of consumption are perfectly correlated so each household also knows aggregate consumption. Since households observe both the return to capital and the aggregate wage, it is straightforward to show that they can recover the aggregate state variables ξ_t.[15]

Thus complete markets reveal complete information, and there is a representative household whose consumption is equal to aggregate consumption,

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[14] The net effect of the payoffs on these securities, if they existed, would be to replace the left-hand side of (2) with a constant share of aggregate income.

[15] The information set is “instantaneously invertible” (Baxter, Graham and Wright, 2008): ie using only t-dated information.

which is a function only of the aggregate states:

$$c_t^* = \eta_\xi^{*\prime} \xi_t \tag{18}$$

where η_ξ^* is a vector of time-invariant coefficients (derived in Appendix B) that can be found by standard solution techniques for rational expectations models.[16]

4.2 Full information and incomplete markets

In this second special case we revert to our central assumption that the only asset available to agents is capital, so agents are unable to trade away idiosyncratic risk. However, despite the absence of complete markets that reveal the idiosyncratic states as in the first benchmark case, it is helpful to consider the case where agents have full information provided as an endowment. We show later (Proposition 3) that this assumption of incomplete markets and complete information is fundamentally inconsistent, but it nonetheless provides a useful analytical building block. This version of the model is essentially a linearised version of that in Krusell and Smith (1998) who assume that agents know the value of aggregates without any account of where this information comes from.

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[16] Maliar and Maliar (2003) show that a complete markets economy with a closely related form of heterogeneity leads to a representative consumer with a utility function with "preference shocks". This does not arise in our economy due to the adding up constraint across idiosyncratic shocks (14), and the multiplicative nature of the shocks (a case noted by Maliar and Maliar, 2003 in their footnote 2).

The properties of this economy are summarised in the following proposition:

Proposition 1 ***(Full information and incomplete markets)*** *With incomplete markets and full information, optimal consumption for a household on island s is*

$$c_t^s|\Omega_t^* = \eta_W^{*\prime} W_t^s = \begin{bmatrix} \eta_\xi^{*\prime} & \eta_\chi^{*\prime} \end{bmatrix} \begin{bmatrix} \xi_t \\ \chi_t^s \end{bmatrix} \tag{19}$$

1. *The coefficients in η_ξ^* are identical to those under complete markets in equation (18).*

2. *The coefficients in η_χ^* solve the undetermined coefficients problem for η_ξ^* in a parallel complete markets economy in which the persistence of aggregate productivity is the same as that of idiosyncratic productivity ($\phi_a = \phi_z$) and the elasticity of intertemporal substitution is zero.*

3. *Aggregate consumption in this economy is identical to the complete markets solution in (18)*

$$\frac{1}{S}\sum_{s=1}^{S} c_t^s|\Omega_t^* = \eta^{*\prime}\xi_t = c_t^* \tag{20}$$

4. *The idiosyncratic element in consumption, $c_t^s - c_t$, is a random walk, and the idiosyncratic element in capital, $k_t^s - k_t$, is a unit root process.*

   Proof. *See Appendix B* ■

The combination of incomplete markets and complete information (provided as an endowment) results in an economy which is identical at an aggregate level to the complete markets economy, but which differs markedly at a household level. The permanent income response to idiosyncratic shocks in turn implies that the idiosyncratic component of consumption is a random walk as in Hall (1978).[17] However, the adding-up constraint across

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[17] Recall Campbell's (1994) result that an economy such as that in the parallel case of Proposition 2b will generate consumption responses in line with the permanent income hypothesis.

idiosyncratic shocks (14) means that such permanent shifts in idiosyncratic consumption cancel out in the aggregate. In general, this form of uninsurable income uncertainty would be expected to cause precautionary saving which would change the steady state of the model, but we preclude these effects by linearizing.

5 Incomplete markets and imperfect information

We now turn to the version of the economy that is central to our analysis, in which we take full account of the informational incompleteness that arises from market incompleteness.

Assumption 1 (market consistent information): *Households only obtain information from the markets they trade in.*

The information set of a household on island s at time t is then

$$\Omega_t^s = \left[\{r_i\}_{i=0}^t , \{v_i^s\}_{i=0}^t , \{k_i^s\}_{i=0}^t , \Xi \right] \tag{21}$$

where Ξ contains the parameters and structure of the underlying model and is therefore time-invariant[18].

We define a measurement vector $i_t^s = \begin{bmatrix} r_t & v_t^s & k_t^s \end{bmatrix}'$ such that the information set evolves according to

$$\Omega_{t+1}^s = \Omega_t^s \cup i_{t+1}^s \tag{23}$$

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[18] Households also have knowledge of the history of their own optimising decisions, defined as

$$\left[\{c_i^s\}_{i=0}^t , \{h_i^s\}_{i=0}^{t-1} \right] \tag{22}$$

however, since each of these histories embodies the household's own responses to the evolution of Ω_t^s, it contains no information not already in Ω_t^s.

In Appendix A we show that the first two observables are given by

$$
\begin{aligned}
r_t &= \lambda\left(a_t + h_t - k_t\right) && (24)\\
v_t^s &= v_t + z_t^s && (25)
\end{aligned}
$$

while the third, k_t^s, can be trivially expressed in terms of the first and third elements of W_t^s, as defined after (16). After substituting for h_t and v_t in (24) and (25) we have

$$i_t^s = H_W' W_t^s + H_c c_t \tag{26}$$

where the matrices H_W and H_c are defined in Appendix A. This information set does not, in general, allow households to recover either aggregate or idiosyncratic states.

The informational problem in our model arises because, since aggregates are not directly observable, households are unable to distinguish between aggregate and idiosyncratic productivity shocks. We shall show that they will therefore make errors (and will know that they must make errors) in estimating the true values of the states. Thus innovations in the observable variables could be caused either by true innovations to the exogenous processes, or by households' estimates of the aggregate states being incorrect. But the informational problem for each household is not restricted to forming estimates of the states, W_t^s, since each household knows that the aggregate capital stock depends on the average expectation of these states. This in turn must imply that the average expectation of the average expectation also matters, and so on - hence the true problem has an infinite dimensional state vector

5.1 The hierarchy of expectations

The state vector relevant to household on island s, X_t^s can be shown to consist of the non-expectational states W_t^s, defined after (16), and an infinite hierarchy of average expectations of W_t^s (Townsend, 1983, Woodford, 2003, Nimark, 2007, 2008)[19]

$$X_t^s = \begin{bmatrix} W_t^s & W_t^{(1)} & W_t^{(2)} & W_t^{(3)} & .... \end{bmatrix}' \tag{27}$$

where the first-order average expectation $W_t^{(1)}$ is an average over all households' expectations of their non-expectational state vector

$$W_t^{(1)} = \frac{1}{S}\sum_{s=1}^{S} E_t^s W_t^s \tag{28}$$

and higher-order expectations are given by

$$W_t^{(k)} = \frac{1}{S}\sum_{s=1}^{S} E_t^s W_t^{(k-1)}; k > 1 \tag{29}$$

The consumption of a household on island s is then

$$c_t^s = \eta' E_t^s X_t^s \tag{30}$$

where η is a vector of coefficients that satisfy the Euler equation (15). The definition of aggregate consumption (10) implies

$$c_t = \eta' X_t^{(1)} \tag{31}$$

where

$$X_t^{(1)} = \frac{1}{S}\sum_{s=1}^{S} E_t^s X_t^s = \begin{bmatrix} W_t^{(1)} & W_t^{(2)} & W_t^{(3)} & .... \end{bmatrix}' \tag{32}$$

5.2 The household's signal extraction problem

To implement optimal consumption (30), a typical household on island s must form estimates of the state vector X_t^s by using the information Ω_t^s available to it. The optimal linear filter is the Kalman filter, however this

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[19] In Appendix C we provide a heuristic argument that demonstrates how the hierarchy of expectations arises from the problem of infinite regress; we also illustrate why the Law of Iterated Expectations cannot be used to simplify this problem.

problem differs from the standard Kalman filter in two ways. The first difference is that the states depend on the household's choice variable c_t^s. Baxter, Graham and Wright (2008) describe this "endogenous" Kalman filter in detail, and give conditions for its stability and convergence which are satisfied here. Secondly, since the aggregate states depend on aggregate consumption, and hence the behaviour of all other households, we need to make an assumption about what a household on one island knows about the behaviour of households on all other islands. We follow Nimark (2007) in assuming that each household applies the Kalman Filter to the entire model on the assumption that each other household is behaving in the same way.

Assumption 2: *It is common knowledge that all households' expectations are rational (model consistent).*

Nimark (2007) discusses this assumption in more detail, but it is essentially a generalization of the full information rational expectations assumption given idiosyncratic information sets.

Given Assumption 2, we show in Appendix D that each household faces a symmetric endogenous Kalman filter problem of the form

$$\begin{aligned} X_{t+1}^s &= Lc_t^s + MX_t^s + Nu_{t+1}^s & (33) \\ i_t^s &= H'X_t^s & (34) \end{aligned}$$

where L, M, N and H are matrices yet to be determined, u_t^s is the innovation in (16) and i_t^s is the measurement vector of household s, defined before equation (23).

Proposition 2 ***(Equilibrium with market-consistent imperfect information)*** *In an economy in which each household*

a. has an information set of the form (21)

b. forms optimal forecasts of the states X_t^s by solving the household-specific filtering problem given by (33) and (34)

c. chooses consumption to satisfy its Euler equation (15) *an equilibrium which satisfies Assumption 1 and Definition 1 is a fixed point of the following undetermined coefficients problem:*

$$\begin{aligned}
M &= \left\{ \begin{bmatrix} F_W & F_c\eta' \\ 0_{\infty,r} & L\eta' + (I - \beta H')\, M \end{bmatrix} + \begin{bmatrix} 0_{r,\infty} \\ \beta H' M T_X \end{bmatrix} \right\} & (35) \\
N &= \begin{bmatrix} I_r \\ \beta H N T_u \end{bmatrix} & (36) \\
\eta' &= (\eta' - \mu')\,[M + L\eta'] & (37)
\end{aligned}$$

where β, the gain matrix of the endogenous Kalman filter, is defined in Appendix D and shows how the measured variables update the state estimates

$$E_t^s X_t^s - E_{t-1}^s X_{t-1}^s = \beta\left(i_t^s - E_t^s i_t^s\right) \quad (38)$$

and L ,H, T_X, T_u and μ are defined in Appendix D.

Proof. See Appendix D ■

5.3 Solution technique

We solve the iterative system of equations given by (35) to (37) for a typical household. The solution to this problem implies a law of motion both for any individual household's state estimates, which evolve by (38), but also, when we average across such updating rules, for the hierarchy of average expectations. This in turn, via (31), determines the solution for aggregate consumption, consistent with each household solving a symmetric filtering and optimal consumption problem. While we model the behaviour of a typical household, there is no representative household in this economy since there is no household whose behaviour represents the aggregate economy.

6 Properties of the economy with incomplete markets and imperfect information

In this section we derive analytical results which show that imperfect information changes the nature of the economy, and explain the mechanism behind this. To simplify the analysis, the propositions in this section consider only the case of fixed labour supply ($\gamma = \infty$).[20]

Proposition 3 ***(Non-Replication of Full Information)*** *If the variance of the idiosyncratic shocks is non-zero ($\sigma_z > 0$), the economy described in Proposition 2 can never replicate the full information economy. However deviations from full information are transitory even when there are permanent shocks to aggregate technology ($\phi_a = 1$)*

Proof. See Appendix E ■

In the economy we describe, households have a restricted information set, given by (21). Proposition 3 shows that this informational problem always matters for the equilibrium of the economy. This result is non-trivial, since it contradicts the simple intuition that arises from counting shocks and signals. Each household needs to identify two underlying shocks, ϖ_t and ϖ_t^s, and observes two "signals", their (idiosyncratic) wage and the (aggregate) return on capital. Since the latter is only affected by the aggregate shock, ϖ_t, it might appear that ϖ_t could be recovered simply from the history of returns. The proof shows that there is indeed a fixed point of the problem in Proposition 2 in which this happens; but that this fixed point is unstable.[21] Thus full information cannot be replicated, except in the following limiting case:

Corollary 1 *As the economy approaches the limiting homogeneous case (as $\sigma_z \to 0$) it approaches the full information economy. Furthermore, in this*

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[20] We conjecture that all remain valid under variable labour supply. This can be verified numerically, but the analytical proofs become much more convoluted. In the proofs we note the implications of relaxing the assumption.

[21] In time series terms there is a reduced form ARMA representation of r_t in which ϖ_t is the innovation, but this representation is "non-fundamental", and hence ϖ_t cannot be recovered from the history of r_t.

limiting case, as $t \to \infty$ the entire history of returns $\{r_s\}_{s=1}^{t}$ becomes informationally redundant.

In the limit, with no idiosyncratic shocks, all households are, and know themselves to be, identical. Market incompleteness only matters to the extent that households differ from each other, so becomes unimportant as the idiosyncratic shocks disappear, as do the associated informational problems. In the limit each household can perfectly observe both the aggregate wage and the return, and thereby trivially infer the values of the aggregate states. But the corollary goes further than this: given a sufficiently large number of observations, households do not even need the history of returns: an information set consisting only of the history of aggregate wages is sufficient to reveal the states[22].

6.1 The impact of aggregate productivity shocks

We have shown that in general the economy with imperfect information must differ from the full information economy. Since the adding-up constraint across idiosyncratic shocks (14) means that the aggregate economy is only driven by the process for aggregate productivity, the differences from full information must arise from a different dynamic response to aggregate productivity shocks. The following proposition states the key features of this response.

Proposition 4 ***(Impact effects of aggregate productivity shocks)*** *In the economy characterized by Proposition 3, a positive aggregate productivity shock has the following effects on impact:*

a) Household estimates of aggregate capital unambiguously fall;

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[22] In this limiting case the information set consisting just of the entire history of the wage would be "asymptotically invertible". Baxter Graham and Wright (2008) show that the conditions for asymptotic invertibility are closely related to econometric conditions for invertibility of vector autoregressions in Fernandez-Villaverde et al (2007). Note that there is an important contrast here between the informational value of the history of wages and the history of returns.

b) There exists a threshold value of the persistence of idiosyncratic productivity $\bar{\phi}_z > \phi_a$ such that, if $\phi_z < \bar{\phi}_z$, household (and hence aggregate) consumption is unambiguously lower than under full information[23].

Proof. See Appendix F. ■

In our economy households must base their estimates of non-expectational states on the signals they observe from markets. When a positive aggregate productivity shock hits, each household will observe this as a simultaneous rise in the aggregate return on capital and their own wage. While the former is a "pure" signal of aggregates, the latter also contains information on idiosyncratic states. As such it can be interpreted as a “noisy” signal of the aggregate economy, although it differs from standard signal-noise problems in that here what is noise with respect to the aggregate economy conveys information about idiosyncratic states that is also important to the household.

To see why the estimate of capital must fall, recall that a general property of optimal filtering is that forecasts of states must always have lower variance than the actual states.[24] This implies that the household's estimate of aggregate productivity must respond less to shocks than does actual productivity i.e. if we start from a steady state in $t = 0$, $E_1^s a_1 < a_1$. But estimates must be consistent with the information set, i.e. $E_1^s r_1 = r_1 = \lambda(a_1 - k_1)$, from (24), so

$$a_1 - k_1 = E_1^s(a_1 - k_1) \tag{39}$$

Since capital is predetermined, $k_1 = 0$ so

$$E_1^s k_1 = E_1^s a_1 - a_1 < 0 \tag{40}$$

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[23] We show numerically in Section 7.5 that $\bar{\phi}_z$, the upper bound for ϕ_z, the persistence of idiosyncratic productivity, is always very close to unity, so this is very close to being a general result.

[24] For some variable q_t and household s's estimate thereof, $E_t^s q_t$ we can write

$$q_t = E_t^s q_t + f_t^s$$

where f_t^s is a filtering error. Efficiency of the filter implies $cov(q_t, f_t^s) = 0$ so

$$var(q_t) \geq var(E_t^s q_t)$$

Thus each household's estimate of capital (and hence the average estimate) must fall on the impact of a positive innovation to aggregate productivity. What is unambiguously good news under full information appears, under incomplete markets, to be a mixture of good and bad news.

The nature of the consumption response, as given in part b) of the proposition is also driven by the requirement that state estimates are consistent with observations. Households know their own capital, k_t^s, which is predetermined. This implies that if a household revises its estimate of aggregate capital downwards, it must revise its estimate of the idiosyncratic component of its own capital $k_t^s - k_t$ *upwards* by exactly the same amount. It is also quite easy to show (see Appendix E.2.2) that the same must apply for the estimate of idiosyncratic productivity. Thus what appears to be bad news on capital in the aggregate economy is always offset by good news on the idiosyncratic economy.

As idiosyncratic productivity becomes more persistent, an estimated positive innovation to idiosyncratic productivity becomes better news. But the parameter restriction in part b) of Proposition 4 states that, unless the persistence of idiosyncratic productivity becomes very high, the bad news about aggregate capital will always outweigh the good news on the idiosyncratic economy. Since aggregate shocks affect all households symmetrically (though not observably so), this implies that the response of aggregate consumption must also be strictly less than under full information.

7 The response of consumption to aggregate productivity shocks

In this section we calibrate our model economy and show that the response of aggregate consumption to productivity shocks under incomplete markets and market-consistent information is not only qualitatively but quantitatively significantly different from that under full information. We carry out sensitivity analysis to all of the important parameters in Section 7.5 and show that our result is robust to plausible changes in the calibration.

7.1 Calibration

The key parameters are the persistence and innovation variance of the aggregate and idiosyncratic productivity processes. We calibrate the aggregate productivity shock with the benchmark RBC values for persistence of $\phi_a = 0.9$ and an innovation standard deviation $\sigma_a = 0.7\%$ per quarter (Prescott, 1986). In Appendix G we discuss the details of our calibration of the idiosyncratic technology process, drawing on the empirical literature on labour income processes. A calibration that sets idiosyncratic persistence equal to aggregate persistence (i.e. $\phi_z = \phi_a = 0.9$) appears consistent with Guvenen's (2005, 2007) recent estimates using US panel data. There is however strong evidence that idiosyncratic technology has a much higher innovation standard deviation. In Appendix G we show that a figure of 4.9% per quarter is consistent with Guvenen's results[25].

Card (1994) estimates the intertemporal elasticity of labour supply, $\frac{1}{\gamma}$ to be between 0.05 and 0.5. For our baseline calibration, we choose $\gamma = 5$, in the middle of this range. For the other parameters we follow Campbell (1994)[26].

7.2 Numerical solution method

All of our theoretical results relate to a representation with an infinite dimension state vector. We follow Nimark (2007) by truncating the hierarchy and writing a state vector of the form

$$\bar{X}_t^s = \left[\begin{array}{cccc} W_t^s & W_t^{(1)} & .... & W_t^{(h)} \end{array} \right]' \tag{41}$$

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[25] Our calibration technique takes account of households' observing their own labour supply, but it can be argued that some idiosyncratic innovations to labour productivity may also be directly observable. We address this issue in Section 7.5.

[26] $\delta = 0.025; \alpha = 0.667; \beta = 0.99; g = 0.005$. We take steady state labour is $H = 0.33$ which implies the weight of labour in the utility function is $\theta = 3.5$.

where h is the order of the truncation[27]. For our baseline calibration, we use $h = 5$. Adding an extra order to the hierarchy would change the impact effect of consumption reported below by 10^{-7}.

7.3 The nature of impulse response functions

The response profiles discussed in this section differ from standard impulse response functions under full information, in that we examine the impact of a shock to an underlying stochastic process, a_t, that would be unobservable to any agents in the economy. The impulse response functions we obtain could not therefore be observed contemporaneously.

As a result of this informational asymmetry between agents and the observer, the stochastic properties of the model are crucial in determining the nature of impulse response functions, in a way that they are not under full information. Under full information, after the initial shock has taken place, the remainder of the impulse response is equivalent to a perfect foresight path, and is thus known in advance to both observer and agents in the model. Furthermore, given the linearity of the model, the entire history of the economy can be split into a sequence of impulse responses to each individual shock. In contrast, in our economy, the agents in the model are continuously making inferences from new information as it emerges, and thus are uncertain not only about the value of future shocks, but also about their own future behaviour in response to past shocks. In making these inferences the underlying stochastic properties of the model are crucial, in a way that they are not under full information.[28]

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[27] Nimark (2007) shows that when the states are exogenous the infinite hierarchy can always be approximated to an arbitrary accuracy by a finite order representation. Our numerical results show that this is also true in our model with endogenous states.

[28] Impulse responses under incomplete information depend on the parameters in the true covariance matrix of structural shocks. Under full information they do not. Also, it is not the functional form of impulse responses, but the shocks that feed into them, that are unobservable. The assumption of common knowledge of rationality means that any household could draw Figure 1, but no household would be able to identify contemporaneously that a productivity shock had actually occurred.

7.4 Response to an aggregate productivity shock

Figure 1 shows the effect of a 1% positive innovation in the process for aggregate productivity on aggregate consumption in our baseline model and in the case of full information. Under full information consumption increases on impact. With incomplete markets and market-consistent information, the response of aggregate consumption is significantly negative on impact of a positive productivity shock.

[FIGURE 1 ABOUT HERE]

This response can be understood in the context of Propositions 1 and 4. Households do not observe the aggregate technology shock directly, but only the associated positive innovations to the aggregate return and the idiosyncratic wage. They then use these observed innovations to updates their estimates of the states according to (38) and it is these state estimates which determine their consumption decision. Innovations to the observed variables can occur either because of structural innovations, or because households' past state estimates were incorrect. For example, if there is a positive innovation to the return, this could either be caused by a positive aggregate technology shock, or because households had overestimated aggregate capital in the past.

On impact, households interpret what is actually good news about aggregate productivity as bad news about the aggregate capital stock and hence lower their estimate of aggregate capital. This in turn means technology must have been overestimated too[29] which mutes the increase in households estimate of aggregate technology. The response of the state estimates in our base case are shown in Figure 2: this shows that the predominantly bad news about the aggregate economy must be offset by good news on both idiosyncratic technology, and the idiosyncratic component of capital, $k_t^s - k$, since k_t^s itself is observed. But the pure permanent income response to estimates of

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[29] To see this recall that estimates must be consistent with the information set so $r_{t-1} = E_{t-1}^s r_{t-1}$ hence $a_{t-1} - E_{t-1}^s a_{t-1} = k_{t-1} - E_{t-1}^s k_{t-1}$. If the household now believes $E_{t-1}^s k_{t-1}$ was too high this implies $E_{t-1}^s a_{t-1}$ was also too high. Since technology is persistent this will cause $E_t^s a_t$ to fall.

idiosyncratic states that arises from Proposition 1 is small, and is dominated by the response to the estimate of aggregate capital.[30]

[FIGURE 2 ABOUT HERE]

7.4.1 How well do households estimate aggregate states?

Households in our model base their consumption decisions on estimates of the state variables, and the previous sections show that this changes the dynamic response of the economy to productivity shocks. The accuracy of these estimates can be assessed by the covariance matrix of one-step ahead forecasts of the states.[31] For our baseline calibration, under full information the quarterly standard deviations of one-step ahead forecast errors for aggregate technology and aggregate capital would be 0.7% and zero respectively (since capital is pre-determined). In our base case with incomplete information the corresponding figures increase to 1.6% and 2.2%. It is striking that what seems to be a quite modest degree of uncertainty about the true value of the capital stock should be enough to cause such a significant change in the dynamics of the system, especially so, given that recent debates about the true size of the capital stock (see, for example, Hall 2000 or the discussion of intangibles in Laitner and Stolyarov, 2003) have suggested measurement errors by statistical offices that are many orders of magnitude larger than this. The relative accuracy of households' estimates in our simple model suggests we may well be considerably understating the informational problem households face.

7.5 Sensitivities

How robust is the negative impact response of consumption to changes in the calibration? The informational problem which drives our results is about

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[30] The heuristic discussion above is implicitly framed in "certainty-equivalent" terms, ie it assumes that the typical household's consumption only depends on estimates of the underlying states, W_t^s, whereas we showed in Section 5.1 that in principle consumption also depends on the entire hierarchy of expectations, via (30). However, the proof of Proposition 4 shows that the *sign* of the actual consumption response is determined by the certainty-equivalent response.

[31] Given by the matrix P, defined by (D.16) in the Appendix.

identifying which shock has occurred, so it is the parameters of the two exogenous processes which have the greatest impact on our results. The standard real business cycle parameters do not have any great effect, since they do not change the nature of the informational problem[32].

[TABLE 1 ABOUT HERE]

Proposition 4, part b) states that there is a threshold value $\bar{\phi}_z$ of the persistence of the idiosyncratic shock, such that, for lower values of ϕ_z the impact response is less than that under full information. Table 1 shows this threshold, both for the fixed labour supply case considered in the proposition and the calibrated value of γ, for different values of the persistence of aggregate technology. It is always very close to unity, so Proposition 4b is very close to being a general result: consumption virtually always under-responds when compared with the full information case.

[TABLE 2 ABOUT HERE]

Table 2 shows how the impact response of consumption to a true aggregate productivity shock varies with the persistence of the aggregate shock, ϕ_a and that of the idiosyncratic shock, ϕ_z (the baseline calibration is in bold). Unconditional variances determine the signal extraction problem, so as the persistences fall, so too does the degree of the informational problem and the response of consumption becomes less negative. However for the idiosyncratic process, there is an offsetting effect As the idiosyncratic shock becomes more persistent, the "good news" from an estimated innovation to idiosyncratic productivity offsets the "bad news" on aggregate capital, so the response of consumption becomes less negative. As ϕ_z approaches the critical value $\bar{\phi}_z$ in table 2 the response of consumption becomes less negative.

[TABLE 3 ABOUT HERE]

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[32] In Appendix H we show that a higher elasticity of labour supply, $\frac{1}{\gamma}$ can significantly amplify our results, but does not change them in qualitative terms.

Table 3 shows how the impact response of consumption to a true aggregate productivity shock varies as the innovation standard deviation σ_z and persistence of the idiosyncratic process (ϕ_z) change. Moving from left to right across the table, we progressively reduce the degree of heterogeneity in the economy, with the final column being the limiting homogenous case of Corollary 1. As the relative standard deviation of the idiosyncratic shock decreases, the information problem becomes less acute, so the impact response of consumption becomes less negative. As the persistence of the shock falls, the unconditional variance falls so the informational problem also becomes less acute. However this is offset by the second effect described above. Since the unconditional variance is a multiple of the innovation variance, the relative strength of the first effect depends on the magnitude of the innovation variance. For high values of the innovation variance the second effect is dominant. For values in the middle of the variance range, the first effect dominates for low values of persistence, and the second effect for high values.

Tables 2 and 3 show that our result appears robust to a wide range of calibrations of the productivity shocks. Only for relatively non-persistent idiosyncratic shocks, with standard deviations around five times lower than those estimated in the literature, does the impact response of consumption come close to that under full information.

While such low levels of implied variability of idiosyncratic productivity appear far out of line with the empirical evidence from Guvenen (2005,2007) cited above, it is at least possible that some proportion of the empirical variability may be due to strictly idiosyncratic factors that are directly observable to the household, such as illness. Indeed, in the limiting case that such idiosyncratic shocks were perfectly observable, the informational problem would of course entirely disappear. Thus it is not heterogeneity as such, but the degree of unobservable heterogeneity that is crucial to our results. We are not, however, aware of any existing empirical evidence that allows such a direct identification for labour productivity.

7.6 A noisy public signal

Our assumption of market-consistent information is a strong one that allowed us to "inspect the mechanism" of a model with imperfect information. However markets are not the only source of information available to households: government statistical offices and the private sector provide estimates of aggregates. To the extent that these estimates contain information, they will reduce the informational problem and bring the response of consumption closer to the full information case. In this section we introduce such a signal to see how it changes our results.

We extend our measurement vector (34) to include public signal of output which differs from true output by a white-noise error[33]. Figure 3 show how this signal affects the response of aggregate consumption in our model with noise in the public signal with a standard deviation ranging from 1% to 3%.

[FIGURE 3 ABOUT HERE]

Recall that without a public signal (Figure 1) the impact response of consumption was negative. With a standard deviation of the noise in the output measure at the top of the range, the impact response becomes very close to zero. As the accuracy of the signal increases, the response of consumption approaches the full information case.

Although there is currently a lively debate on the empirical effect of technology shocks, see for example Christiano et al (2003), there seems to be some agreement that a range of variables, including consumption, respond more sluggishly in the data than in a standard RBC model. Theoretical explanations for such sluggishness (for example Francis and Ramey, 2005) are usually couched in terms of nominal or real rigidities, or habit formation. The result of this section shows that informational imperfections can generate such a sluggish response of consumption without additional rigidities.

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[33] How noisy are real-time estimates of output? Orphanides and Norden (2002) attempt to quantify the extent of uncertainty by calculating the difference between real-time and final estimates. Their table 2 shows standard deviations of the difference ranging from 1% to 3% per quarter. However, they note that their method "...overestimate[s] the true reliability of the real time estimates since it ignores the estimation error in the final series", which given the issues involved in measuring output, is likely to be large but is by its nature unquantifiable.

7.7 Financial markets

We have seen that our results depend on the combination of (unobservable) heterogeneity and incomplete markets. An obvious question is whether the extreme degree of incompleteness that we have in our model, with physical capital as the only store of value, has significantly affected these results. We have investigated a modified version of the model in which we include the two benchmark "aggregate" financial assets: a risk-free asset and the market portfolio.

In the standard complete markets framework, in which financial assets are simply priced with reference to the Euler Equation of the single representative consumer, the explicit inclusion or exclusion of financial assets into the model makes no difference to the solution. In our incomplete markets framework this is no longer the case: to the extent that financial prices and the observable flows that they price have informational content they will change the equilibrium (indeed we have already seen that if these span the distribution of idiosyncratic shocks we have complete markets and complete information). However the two aggregate assets we consider turn out to modify the equilibrium to only a very limited extent.

To illustrate, consider the case of a risk-free bond. For any individual household, if we apply the linearised Euler equation (15) to the (nonstochastic) risk-free rate, r_{ft} we have

$$E_t^s \Delta c_{t+1}^s = r_{ft} = E_t^s r_{t+1} \tag{42}$$

Since all households optimise their expected consumption growth to the same observable risk-free return, by averaging across households, the risk-free rate makes a linear combination of elements of the hierarchy of average expectations observable. After some manipulation, the risk-free return can be written in terms of first-order average expectations[34] as

$$r_{ft} = \mu' M X_t^{(1)} \tag{43}$$

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[34] A full derivation can be found in Appendix I

where μ is the same vector that relates the return on capital, r_t to the non-expectational aggregate states, defined by (D.18), and M is defined in (35).

When we add this variable to the vector of observables, however, we find barely any change in the form of the impulse responses reported in the previous section. The intuition for this is as follows. In our calibration, with its large variance of the idiosyncratic shock, the wage conveys essentially no useful information about aggregates. Households have a single common signal about aggregates in the return on capital, and know that this is the case: as a result even without a risk-free asset the average expectation of aggregate states is close to being common knowledge. Hence the additional information on the hierarchy that the risk-free rate conveys, via (43) very little additional information, and thus the equilibrium is little changed. We find very similar results when a stock market is introduced, with a dividend process given by the one period return on capital (as in Lettau, 2003).[35]

The relative unimportance of these two aggregate assets for the solution of our economy has an interesting parallel with the results of Athanasoulis and Shiller (2000), who find that, in a model of missing markets where new asset markets are added incrementally according to the magnitude of the resulting increase in welfare, the last asset to be added is the market portfolio itself, and that all preceding assets that are added are pure swaps, since these provide the crucial rule of risk pooling. In our framework, similarly, the assets that would provide the crucial information would be those that span the distribution of idiosyncratic shocks and thereby allow risk sharing via swaps. Aggregate markets thus appear to play a marginal role both in terms of welfare and in terms of information.

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[35] The forward pricing equation for the stock market is complicated by the need to allow for the hierarchy of expectations: we cannot simply apply the law of iterated expectations in solving forward, but instead need to substitute out for increasingly higher orders of average expectations as we move forward (along very similar lines to the methodology used by Nimark, 2007). But ultimately we can derive a closed form expression for the stock price, and hence the return, in terms of the full hierarchy, which for numerical solutions we can again truncate as outlined in Section 7.2. Precise details of the methodology (which we aim to apply in future work on the properties of asset returns in our model economy) are available from the authors.

7.8 Moments and Correlations

In this section we compare some basic descriptive statistics of our economy with those of the benchmark stochastic growth model which is its full information counterpart. We have shown that there are some striking differences in impact and short-run responses. But we also have shown, in Proposition 2, that differences from the full information equilibrium must always be transitory. Thus differences in *un*conditional moments and correlations are, as Table 4 shows, in general much more modest than might be suggested by simply looking at impulse responses.

The model with no public signal has a significantly reduced correlation between consumption and output (driven by the negative impact response of technology shocks) but the correlation remains strongly positive, since as Figure 1 shows, beyond the first few quarters the response of consumption is positive, and hence of the same sign as output. While the correlation in this version of the model is lower than the value of 0.8 typically estimated in the data (e.g. Cooley, 1995, p30), the inclusion of a public signal, even a poor one, raises the correlation, and it is easy to see how the inclusion of other features, for example credit-constrained consumers, could raise it further.

The most striking difference between the models is the negative correlation between consumption and the return on capital in the incomplete information model, as compared to a correlation close to zero with full information. A very similar correlation is found between consumption and the risk-free rate in the extended version of the model discussed in the previous section. We think this is potentially interesting from an asset pricing perspective. Lettau (2003) demonstrates that the standard full information model can quite often generate negative equity premia because consumption and one period returns are positively correlated under full information, which means that for quite a range of calibrations prices of multiperiod assets like bonds or equity should fall on impact of a productivity shock, via the discounting of cashflows. The negative correlation of such asset prices with consumption gives them hedge value for the representative consumer under full information. Once the correlation between one period returns and consumption is negative, as in our model, this at least opens up the possibility of positive equity premia. We aim to investigate this issue further in future research.

8 Conclusions

We believe that our model is only a starting-point for the analysis of the link between heterogeneity, market incompleteness and informational problems. We have shown a very stark contrast with the standard complete markets model, and with an incomplete markets model in which full information is simply assumed. This has particular implications for the growing number of incomplete markets models which follow Krusell and Smith (1998) in adopting a solution method that is based on the assumption that the aggregate states, typically the capital stock, are observable.

We do not yet know how robust this contrast will be to further modifications. On the one hand it might be argued that we are overstating the informational implication of incomplete markets. Evidently markets are not so incomplete as in our model. While we have shown that aggregate financial markets make very little difference to our results, the existence of insurance and other risk-pooling markets would push our results closer to those under full information. We have also noted that it is not heterogeneity as such that drives our results for the aggregate economy, but *unobservable* heterogeneity. This suggests a possible field for future empirical research.

On the other hand, it is very easy to put forward arguments that we may be significantly understating the extent of the informational problem. Our model is highly simplified, with only a single source of idiosyncratic uncertainty; symmetry across households; and a single aggregate endogenous state variable. More realistic models will have more shocks and more states (for example Smets and Wouters, 2007, has seven shocks and four states) which will make the signal-extraction problem of the household much more difficult, and also more sources of information. An important direction for future work is to use our techniques in such a model, which would enable us to draw more robust conclusions. But we have noted that a striking feature of our model is how *well* households can estimate the capital stock, in stark contrast to the observed wide variation in estimates within the academic debate (e.g., Hall, 2001, Laitner and Stolyarov, 2003), which, it might be argued, reflect the much greater complexity of the true inference problem.

We also assume that agents know the structure and parameters of our model. There is a large body of research, both on model uncertainty (for example Hansen and Sargent, 2001) and learning (Evans and Honkapohja, 2003) that would question these assumptions. In the context of our model, a natural question to ask is whether the joint time series process for the observables that arises from the solution to the filtering problem is learnable; and even if it is, whether sufficient identifying assumptions can be made to be able to derive the underlying structural parameters of the model (a potentially significant inferential problem which we simply assume away) Even if both these strong conditions are met, it is easy to see that the inferences made by agents in our model would require very large amounts of data.

Until these issues have been investigated, we would hesitate to draw strong empirical conclusions from our analysis. Nonetheless our results are in distinct contrast to the standard benchmark model in weakening the positive short-term correlation between consumption, output, and the return on capital implied by full information. We suspect that the alternative dynamics implied by our analysis may generate insights into the well-known puzzles in macroeconomics and finance relating to these correlations.

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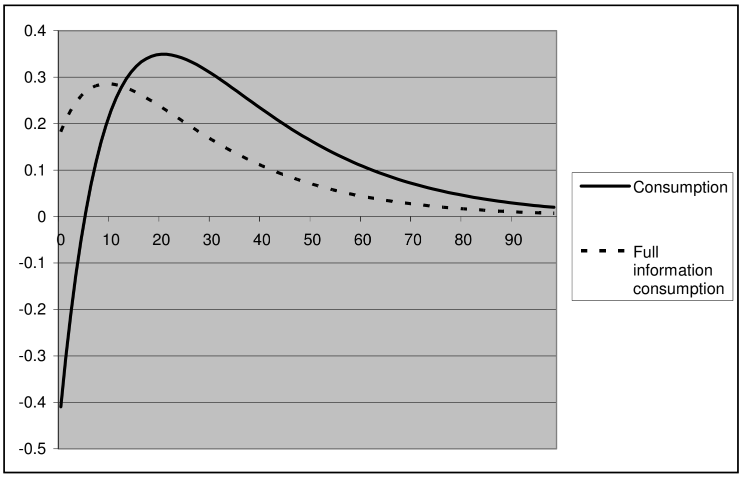
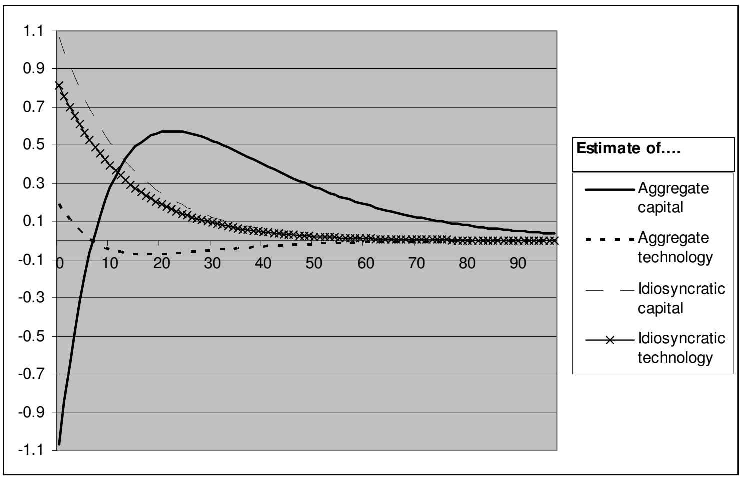
Figures and tables

Figure 1: Response of consumption to a 1% positive innovation to aggregate productivity



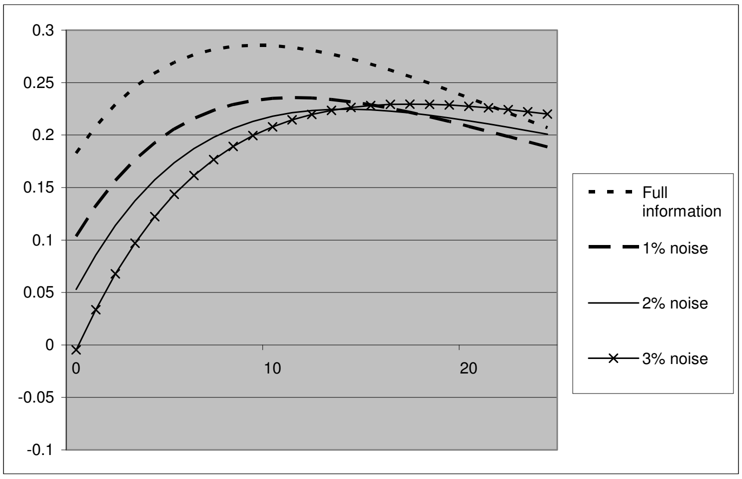
x-axis shows periods; y-axis shows percentage deviations from steady state

Figure 2: Response of state estimates to a 1% positive innovation to aggregate productivity



x-axis shows periods; y-axis shows percentage deviations from steady state

Figure 3: Response of aggregate consumption to a 1% positive innovation to aggregate productivity with a noisy public signal of output



x-axis shows periods; y-axis shows percentage deviations from steady state

Table 1: Critical values, $\bar{\phi}_z$ (as defined in Proposition 4) of persistence of idiosyncratic shock

ϕ_a	0.95	0.9	0.8	0.5	0.2	0
Fixed labour: $\gamma = \infty$	0.998	0.997	0.995	0.994	0.993	0.993
Variable labour:$\gamma = 5$	0.997	**0.996**	0.995	0.993	0.993	0.992

Base case shown in bold

Table 2: Impact effect of aggregate technology shock on aggregate consumption: sensitivity to persistence parameters

	ϕ_a				
ϕ_z	0.95	0.9	0.85	0.7	0.5
0.95	-0.541	-0.338	-0.238	-0.115	-0.059
0.9	-0.614	$\mathbf{-0.410}$	-0.301	-0.157	-0.087
0.85	-0.603	-0.423	-0.318	-0.172	-0.098
0.7	-0.426	-0.374	-0.305	-0.180	-0.107
0.5	-0.097	-0.245	-0.241	-0.169	-0.107

Base case shown in bold

Table 3: Impact effect of aggregate technology shock on aggregate consumption: sensitivity to properties of idiosyncratic shock

	σ_z/σ_a					
ϕ_z	∞	10	5	2	1	0
0.95	-0.352	-0.345	-0.338	-0.273	-0.113	0.183
0.9	-0.440	-0.425	**-0.410**	-0.276	0.022	0.183
0.85	-0.474	-0.448	-0.424	-0.211	0.058	0.183
0.7	-0.510	-0.438	-0.376	-0.009	0.126	0.183
0.5	-0.526	-0.365	-0.245	0.0763	0.160	0.183

Base case shown in bold

Table 4a: Moments and correlations for full information model

	c	y	a	k	r
Standard deviation (relative to y)	0.63	1.0	0.88	0.93	0.03
Correlations	c	y	a	k	r
c	1	0.83	0.75	0.98	-0.08
y		1	0.99	0.72	-0.49
a			1	0.62	0.60
k				1	-0.25
r					1

Table 4b: Moments and correlations for baseline incomplete information model

	c	y	a	k	r
Standard deviation (relative to y)	0.65	1.0	0.76	1.5	0.04
Correlations	c	y	a	k	r
c	1	0.46	0.08	0.90	-0.97
y		1	0.92	0.80	-0.23
a			1	0.50	0.17
k				1	-0.77
r					1

Table 4c: Moments and correlations for model with noisy output signal

	c	y	a	k	r
Standard deviation (relative to y)	0.56	1.0	0.84	1.5	0.04
Correlations	c	y	a	k	r
c	1	0.76	0.40	0.99	-0.75
y		1	0.90	0.77	-0.15
a			1	0.41	0.30
k				1	-0.75
r					1